THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 67 NUMBER 6

DETROIT PUBLIC LIBRARY
TECHNOLOGY DEPARTMENT

CONTENTS

The Rôle of Compactness in Analysis . . . . . EDWIN HEWITT 499
The Generalized Coconut Problem . . . . . R. B. KIRCHNER 516
On the Brocard Points of a Triangle . . . . . PETER YFF 520
Fibonacci Series Modulo m . . . . . D. D. WALL 525
Mathematical Induction in Commutative Semigroups . . SETH WARNER 533
A Topology for Sequences of Integers II . . . . . R. L. DUNCAN 537
Some Remarks on Arithmetical Identities . . . . . P. J. MCCARTHY 539
Axioms for the Affine Line . . F. CUNNINGHAM, JR. AND C. W. VALENTINE 549
The William Lowell Putnam Mathematical Competition . . L. E. BUSH 559
Mathematical Notes . . . . . NORMAN LEVINE, J. A. NAVARRO, P. M. ANSELONE AND P. PORCELLI, R. BUTLER, FRANK HAHN, L. CARLITZ AND JACK LEVINE, L. E. DUBINS 562
Classroom Notes . . . . R. C. WREDE, H. E. VAUGHAN, J. E. EATON, JACQUELINE P. EVANS, JAMES MICHELOW, JEROME ROTHSTEIN 574
Mathematical Education Notes . . . . . P. S. JONES 585
Elementary Problems and Solutions 593
Advanced Problems and Solutions 596
Recent Publications 601
News and Notices 608
The Mathematical Association of America 610
New Members 610
February Meeting of the Louisiana-Mississippi Section 613
March Meeting of the Southern California Section 615
Officers and Committees as of May 1, 1960 617
Calendar of Future Meetings 620

JUNE-JULY 1960

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

RALPH D. JAMES, *Editor*

ASSOCIATE EDITORS

RICHARD V. ANDREE	G. E. HAY	C. O. OAKLEY
JOHN A. BROWN	L. M. KELLY	J. M. H. OLMSTED
ROY DUBISCH	JOHN R. MAYOR	E. P. STARKE
HOWARD EVES	LLOYD J. MONTZINGO, JR.	H. S. ZUCKERMAN
	IVAN NIVEN	

EDITORIAL CORRESPONDENCE should be addressed to the Editor, R. D. JAMES, Department of Mathematics, University of British Columbia, Vancouver 8, Canada. Articles should be typewritten and double-spaced on 8½ by 11 paper. The greatest possible care should be taken in preparing the manuscript and authors should keep a complete copy.

ADVERTISING CORRESPONDENCE should be addressed to F. R. OLSON, Mathematical Association of America, University of Buffalo, Buffalo 14, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Treasurer, H. M. GEHMAN, Mathematical Association of America, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

OFFICERS OF THE ASSOCIATION

President, C. B. ALLENDOERFER, University of Washington
First Vice-President, A. S. HOUSEHOLDER, Oak Ridge National Laboratory
Second Vice-President, HARLEY FLANDERS, University of California, Berkeley
Secretary, H. L. ALDER, University of California, Davis
Treasurer, H. M. GEHMAN, University of Buffalo
Associate Secretary, LLOYD J. MONTZINGO, JR., University of Buffalo
Editor, R. D. JAMES, University of British Columbia

Additional Members of the Board of Governors: R. C. BUCK, JEWELL H. BUSHEY, LAMBERTO CESARI, R. V. CHURCHILL, H. S. M. COXETER, W. L. DUREN, JR., HOWARD EVES, WILLIAM FELLER, J. S. FRAME, W. T. GUY, JR., G. B. HUFF, W. N. HUFF, J. G. KEMENY, MORRIS KLINE, J. C. KNIPP, P. B. JOHNSON, W. G. LEAVITT, A. T. LONSETH, E. J. MCSHANE, G. M. MERRIMAN, H. T. MUHLY, ARTHUR OLLIVIER, J. M. H. OLMSTED, R. S. PARK, GEORGE PÓLYA, G. B. PRICE, R. G. SMITH, F. M. STEWART, ROTHWELL STEPHENS, R. M. THRALL, W. R. UTZ, JR., R. D. WAGNER, CHARLES WEXLER, ALBERT WILANSKY, C. R. WYLIE, JR., R. C. YATES.

Annual dues for members of the Association (including a subcription to the American Mathematical Monthly) are $5.00. For nonmembers the subscription price is $8.00.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y.
during the months of January, February, March, April, May, June–July,
August–September, October, November, December.

Second-class postage paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Copyright © 1960 by the Mathematical Association of America, Inc.

PRINTED IN THE UNITED STATES OF AMERICA

c
T
r
a
li
a
w
fr
t
a

n
ti

b
o
h

b
o

in
su

e
c
p

e
{
s
e
s

o
ti

c

THE RÔLE OF COMPACTNESS IN ANALYSIS

EDWIN HEWITT, University of Washington*

Introduction. Analysis is an immense field of mathematics, and compactness concepts and arguments enter in a great many different branches of analysis. To give a really adequate picture of the rôle of compactness in analysis would require in fact a survey of much of analysis: a task obviously impossible of accomplishment within the confines of a short essay, to say nothing of the limitations of the writer. After some remarks about the concept of compactness as such, therefore, we shall give examples from various parts of analysis in which compactness enters as a necessary hypothesis. These examples range from highly elementary to fairly sophisticated. They have been chosen partly to illuminate the concept and partly as important theorems of analysis. No attempt has been made to be exhaustive.

Compactness also plays a vital rôle in many existence theorems as a technique of proof. For reasons of space, we must limit ourselves to a brief description of how this is done.

Compact subsets of the line. Compactness for subsets of the real line R has been explicitly recognized, studied, and used for many years. Let A be a subset of the line, and consider the following three properties, which A may or may not have.

(1) Let $\{t_1, t_2, t_3, \cdots, t_n, \cdots\}$ be a sequence of points such that every t_n belongs to A. Then there is a subsequence $\{t_{n_1}, t_{n_2}, t_{n_3}, \cdots, t_{n_k}, \cdots\}$ of the original sequence that converges to a limit t, where t is a point of the set A.

(2) Every infinite subset of A has at least one limit point lying in A.

(3) Let $\{J_\iota\}_{\iota\in I}$ be a family of open intervals such that every point of A lies in at least one of the open intervals J_ι, that is, $\bigcup_{\iota\in I} J_\iota \supset A$. Then there is a finite subfamily $\{J_{\iota_1}, J_{\iota_2}, \cdots, J_{\iota_m}\}$ of $\{J_\iota\}_{\iota\in I}$ such that $J_{\iota_1}\cup J_{\iota_2}\cup \cdots \cup J_{\iota_m}\supset A$.†

It is easy to show, and well known, that properties (1), (2), and (3) are equivalent to each other. A subset of R possessing one, and hence all, of them is called *compact*. It is also elementary and well known that a subset of R is compact if and only if it is closed and bounded.

It is obvious that every finite subset F of R is compact. If F is void, properties (1), (2), and (3) hold trivially. Hence suppose that F is nonvoid. If $\{t_1, t_2, t_3, \cdots, t_n, \cdots\}$ is a sequence such that every t_n lies in F, there must be some point t of F such that $t_n = t$ for an infinite set of positive integers n. (If every $x\in F$ appears as a t_n only a finite number of times, and F is finite, then the sequence $\{t_1, t_2, t_3, \cdots, t_n, \cdots\}$ can have only a finite number of terms.) Let

* Written with financial support from the National Science Foundation. This essay is based on a lecture given several times by the writer while a visiting lecturer of the Mathematical Association of America (March–May 1957).

† The following paraphrase of (3) is attributed to Hermann Weyl (1885–1955). "If a city is compact, it can be guarded by a finite number of arbitrarily near-sighted policemen."

n_1 be the least positive integer n such that $t_n = t$, n_2 the least positive integer n such that $n > n_1$ and $t_n = t$, and so on. The sequence $\{t_{n_1}, t_{n_2}, t_{n_3}, \cdots, t_{n_k}, \cdots\}$ is constant and so has the limit $t \in F$. Property (2) holds vacuously for every finite set $F \subset R$, since F contains no infinite subsets. Property (3) is almost trivial. Let $F = \{x_1, x_2, \cdots, x_p\}$. Then each x_k is in some open interval J_{i_k} $(k = 1, 2, \cdots, p)$, and the finite subfamily $\{J_{i_1}, J_{i_2}, \cdots, J_{i_p}\}$ of $\{J_i\}_{i \in I}$ obviously has the property that $J_{i_1} \cup J_{i_2} \cup \cdots \cup J_{i_p} \supset F$.

Of course there are many *infinite* compact subsets of R. The set $\{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\}$ is such a set. The closed interval $[a, b]$,* for any real numbers a and b such that $a < b$, is also. A curious and instructive example of an infinite compact subset of R is Cantor's ternary set, defined as the set of all numbers of the form $2(\sum_{n=1}^{\infty} x_n 3^{-n})$, where $\{x_1, x_2, x_3, \cdots, x_n, \cdots\}$ is an arbitrary sequence consisting only of 0's and 1's.†

It thus appears that compactness for subsets of R is a generalization of finiteness. The thesis of this essay is that *a great many propositions of analysis are*:

(A) *trivial for finite sets*;
(B) *true and reasonably simple for infinite compact sets*;
(C) *either false or extremely difficult to prove for noncompact sets.*

As we shall see, some care is usually needed in moving from the finite situation to the corresponding compact infinite situation. A given assertion true for finite sets often needs some qualification to be provable for all compact sets.

Example I. As the first illustration of our thesis, consider the following proposition.

I_f. A real-valued function f defined on a (nonvoid) finite set is bounded and attains its least upper bound and greatest lower bound.

This proposition is perfectly obvious. Let f be defined on $\{x_1, x_2, \cdots, x_p\}$ (a subset of R or not). Then the numbers $\max\{f(x_1), f(x_2), \cdots, f(x_p)\}$ and $\min\{f(x_1), f(x_2), \cdots, f(x_p)\}$ are the least upper bound and greatest lower bound, respectively, of the values of f, and are attained by f at points of its domain of definition.

For arbitrary compact subsets of R, we have the following paraphrase of Proposition I_f.

I_c. A continuous real-valued function f defined on a (nonvoid) compact subset of R is bounded and attains its least upper bound and greatest lower bound.

Propositions I_f and I_c illustrate the typical change required in going from the finite to the infinite compact situation. Proposition I_f is true of *all* real-valued

* We use the expression $[a, b]$ to denote the *closed* interval $\{x: x \in R, a \leq x \leq b\}$, and $]a, b[$ to denote the *open* interval $\{x: x \in R, a < x < b\}$. The expressions $[a, b[$ and $]a, b]$ have similar definitions.

† We make no effort here to classify all compact subsets of R.

functions, while Proposition I_c requires that f be continuous. Similar changes are almost always needed in going from the finite to the infinite compact situation.*

Proposition I_c is proved in every carefully written treatment of the elementary theory of functions of a real variable.† Nevertheless we sketch a proof. Let A be a compact nonvoid subset of R and f a continuous real-valued function defined on A. Let x be any point of A. By continuity, there is a number $\delta_x>0$ such that $|f(x)-f(t)|<1$ for all $t\in A$ such that $|t-x|<\delta_x$. The family of open intervals $\{\,]x-\delta_x, x+\delta_x[\,\}_{x\in A}$ thus satisfies the conditions of property (3) above. Hence there are finitely many of these intervals, say $]x_1-\delta_{x_1}, x_1+\delta_{x_1}[$, $]x_2-\delta_{x_2}, x_2+\delta_{x_2}[, \cdots,]x_m-\delta_{x_m}, x_m+\delta_{x_m}[$, whose union contains A. It then follows that $|f(t)|<1+\max\{|f(x_1)|, |f(x_2)|, \cdots, |f(x_m)|\}$ for all $t\in A$: thus f is bounded. To prove that the least upper bound d of the set $\{f(x): x\in A\}$ is a value of f, it is convenient to use property (1) of compact subsets of R. By definition of least upper bound, we can find, for every positive integer n, a point $t_n\in A$ such that $f(t_n)>d-1/n$. By property (1), there is a subsequence $\{t_{n_1}, t_{n_2}, t_{n_3}, \cdots, t_{n_k}, \cdots\}$ of the sequence $\{t_1, t_2, t_3, \cdots, t_n, \cdots\}$ that converges to a limit t, where $t\in A$. The continuity of f then implies that $f(t)=d$. Similarly one shows that the greatest lower bound of the set $\{f(x): x\in A\}$ is a value of f.

Parts (A) and (B) of our thesis are thus demonstrated for the simple propositions I_f and I_c. Part (C) is also true for Proposition I, as the following statement shows.

I_n. Let B be a noncompact subset of R. Then there is an unbounded continuous real-valued function on B, and there is a bounded continuous real-valued function on B that assumes neither its least upper nor greatest lower bound.

If B is noncompact and contained in R, it is either unbounded or nonclosed. If B is unbounded, let $f_1(x)=x$ for $x\in B$ and let $f_2(x)=|x|/(1+|x|)$ for $x\in B$. Clearly f_1 and f_2 are continuous, f_1 is unbounded, and f_2 is bounded and does not attain its least upper bound 1. (It is easy, although a little more complicated, to produce a bounded continuous real-valued function on B assuming neither its least upper nor greatest lower bound: we omit the details.) If B is nonclosed, let u be a limit point of B that does not lie in B. Then let $f_1(x)=1/(x-u)$ for $x\in B$ and $f_2(x)=e^{-|x-u|}$ for $x\in B$. Clearly f_1 and f_2 are continuous, f_1 is unbounded, and f_2 is bounded and does not attain its least upper bound 1.

Example II. We now take up the classical Weierstrass approximation theorem and its neoclassical generalization due to M. H. Stone.‡ For functions on

* Since every real-valued function defined on a finite subset of R is continuous, we can insert "continuous" before "function" in Proposition I_f without altering its force, and thus make the hypotheses on f identical in I_f and I_c. This however seems to be a little artificial.

† See for example [1], p. 73.

‡ Stone's general approximation theorem appeared in [19] as Theorem 82. An extended account of the theorem and its applications appears in [20]. For a somewhat different treatment, see [7].

finite sets, we have as before a simple assertion.

II_f. *Let F be any finite set (a subset of R or not). Let $\mathfrak{S}$ be a set of real-valued functions on F such that for every pair x and y of distinct points in F, there is some function f belonging to the set $\mathfrak{S}$ such that $f(x) \neq f(y)$.* Then every real-valued function on F is equal to a polynomial in functions chosen from the set $\mathfrak{S}$.*

Proposition II_f may appear somewhat artificial, and perhaps it is. Its generalization II_c to the compact situation, however, is an extremely powerful result, which is rightly regarded as one of the landmarks of modern analysis.

We proceed to a proof of II_f. Write the points of F as a finite sequence with no repetitions: $F = \{x_1, x_2, \cdots, x_p\}$. By hypothesis, there is a function f_{1j} in $\mathfrak{S}$ such that $f_{1j}(x_1) \neq f_{1j}(x_j)$, for $j = 2, 3, \cdots, p$. Let φ_1 be the function on F defined by

$$\varphi_1 = \prod_{j=2}^{p} [f_{1j} - f_{1j}(x_j)][f_{1j}(x_1) - f_{1j}(x_j)]^{-1}.$$

Plainly φ_1 is a polynomial in functions from $\mathfrak{S}$. It is also plain that $\varphi_1(x_1) = 1$ and $\varphi_1(x_j) = 0$ for $j = 2, 3, \cdots, p$. In like manner, we construct functions $\varphi_2, \varphi_3, \cdots, \varphi_p$, each a polynomial in functions from $\mathfrak{S}$, such that $\varphi_j(x_j) = 1$ and $\varphi_j(x_k) = 0$ for $k \neq j$ ($j = 1, 2, \cdots, p$; $k = 1, 2, \cdots, j-1, j+1, \cdots, p$). Let g be any real-valued function on F. Then $g = g(x_1)\varphi_1 + g(x_2)\varphi_2 + \cdots + g(x_p)\varphi_p$; this proves Proposition II_f.†

The infinite compact analogue for II_f follows.

II_c. *Let A be an arbitrary compact subset of R, and let $\mathfrak{S}$ be a separating set of continuous real-valued functions on A. Let g be any continuous real-valued function on A and ϵ any positive real number. Then there is a polynomial $p = p(f_1, f_2, \cdots, f_r)$ in functions $f_1, f_2, \cdots, f_r$ taken from $\mathfrak{S}$ such that $|g(x) - p(x)| < \epsilon$ for all $x \in A$. In other words, every continuous real-valued function on A is the uniform limit (on A) of a sequence of polynomials formed from functions of the set $\mathfrak{S}$.*

There are many proofs of Proposition II_c. A basic point in several of these is the fact that the absolute value of any bounded real-valued function ψ on A is the uniform limit of polynomials with real coefficients in the function ψ^2.‡

It follows at once from Raabe's test for the convergence of series of positive terms ([13], p. 285) that the series

$$\sum_{k=0}^{\infty} \left| \binom{\frac{1}{2}}{k} \right|$$

converges. We also have $|t| = (1 + (t^2 - 1))^{1/2}$ for every real number t. Let α be

* Such a set of functions is called a *separating* set of functions, for an obvious reason.

† Proposition II_f is true for functions with values in any division ring; hence for complex- or quaternion-valued functions defined on F.

‡ This is the fundamental idea in Lebesgue's proof [15], which deals with a special case. It is also pointed out explicitly by Stone ([19], p. 467) and is used by Hewitt [7].

the least upper bound of the set $\{|\psi(x)| : x \in A\}$. Then, for every $x \in A$, we have

$$\left| |\psi(x)| - \sum_{k=0}^{n} \alpha \binom{\frac{1}{2}}{k} (\alpha^{-2}\psi^2(x) - 1)^k \right|$$

$$= \alpha \left| |\alpha^{-1}\psi(x)| - \sum_{k=0}^{n} \binom{\frac{1}{2}}{k} (\alpha^{-2}\psi^2(x) - 1)^k \right|$$

$$= \alpha \left| \sum_{k=n+1}^{\infty} \binom{\frac{1}{2}}{k} (\alpha^{-2}\psi^2(x) - 1)^k \right| \leqq \alpha \sum_{k=n+1}^{\infty} \left| \binom{\frac{1}{2}}{k} \right|.$$

Since the last expression is arbitrarily small if n is sufficiently large, the first observation is proved.

We next notice that $\max(a, b) = \frac{1}{2}(|a - b| + (a + b))$ and $\min(a, b) = \frac{1}{2}(-|a-b| + (a+b))$ for real numbers a and b. Hence, if ψ_1 and ψ_2 are any bounded real-valued functions on A, the functions $\max(\psi_1, \psi_2)$ and $\min(\psi_1, \psi_2)$ are uniform limits of sequences of polynomials in the functions ψ_1 and ψ_2. The same is also true, clearly, for $\max(\psi_1, \psi_2, \cdots, \psi_k)$ and $\min(\psi_1, \psi_2, \cdots, \psi_k)$, if $\psi_1, \psi_2, \cdots, \psi_k$ are bounded real-valued functions on A $(k = 2, 3, 4, \cdots)$.*

Now let f be any continuous real-valued function on A. Let

$$c = \min\{f(x) : x \in A\} \quad \text{and} \quad d = \max\{f(x) : x \in A\}.$$

If $c = d$, f is a constant and is trivially a polynomial in functions from $\mathfrak{S}$. If $c < d$, let $g = (f - c)/(d - c)$. Then $\min\{g(x) : x \in A\} = 0$, $\max\{g(x) : x \in A\} = 1$, and if g can be arbitrarily uniformly approximated by polynomials in functions from $\mathfrak{S}$, then f can also be so approximated.

Let $E = \{x : x \in A,\ g(x) \leqq \frac{1}{3}\}$ and $F = \{x : x \in A,\ g(x) \geqq \frac{2}{3}\}$. Let x and y be any points in E and F respectively. Let g_{xy} be a function from the set $\mathfrak{S}$ such that $g_{xy}(x) \neq g_{xy}(y)$. Let h_{xy} be the function

$$h_{xy} = \frac{1}{4} + \frac{1}{2(g_{xy}(y) - g_{xy}(x))} (g_{xy} - g_{xy}(x)).$$

Then $h_{xy}(x) = \frac{1}{4}$ and $h_{xy}(y) = \frac{3}{4}$ and h_{xy} is obviously a polynomial in functions from $\mathfrak{S}$. Now regard y as fixed for a moment and let x vary in E. Since h_{xy} is continuous, there is an open interval I_x containing x such that $h_{xy}(t) < \frac{1}{2}$ for all $t \in I_x \cap A$. It is easy to see that E, as a closed subset of A, is compact. Hence there are a finite number of open intervals $I_{x_1}, I_{x_2}, \cdots, I_{x_m}$ whose union contains E. Let φ_y be the function defined by

$$\varphi_y = \min\{h_{x_1 y}, h_{x_2 y}, \cdots, h_{x_m y}\}.$$

Then φ_y is the uniform limit of polynomials in functions from $\mathfrak{S}$ and has the

* Obviously the last two observations hold for bounded real-valued functions defined on *any* set.

property that $\varphi_y(t) < \frac{1}{3}$ for $t \in E$ and $\varphi_y(y) = \frac{2}{3}$. Now letting y vary over F, using the compactness of F, and forming a maximum, we find a function ψ that is the uniform limit of polynomials in functions from $\mathfrak{S}$ with the properties that $\psi(t) < \frac{1}{3}$ for $t \in E$ and $\psi(t) > \frac{2}{3}$ for $t \in F$. The function

$$\omega_1 = \min\{\max[\tfrac{1}{3}, \psi], \tfrac{2}{3}\}$$

is also the uniform limit of polynomials in functions from $\mathfrak{S}$. It is clear that $\omega_1(t) = \frac{1}{3}$ for $t \in E$, $\omega_1(t) = \frac{2}{3}$ for $t \in F$, and $\frac{1}{3} \leqq \omega_1(t) \leqq \frac{2}{3}$ for all $t \in A$. The function $g_1 = g - \omega_1$ is continuous and has minimum and maximum values $-\frac{1}{3}$ and $\frac{1}{3}$ on A. Applying the foregoing argument to the function $\frac{3}{2}g_1 + \frac{1}{2}$ we find a function ω_2 that is the uniform limit of polynomials in functions from $\mathfrak{S}$ such that $\max\{|\frac{3}{2}g_1(t) + \frac{1}{2} - \omega_2(t)| : t \in A\} = \frac{1}{3}$. Thus we have

$$\max\{|\, g(t) + \tfrac{1}{3} - \omega_1(t) - \tfrac{1}{2}\omega_2(t)\,| : t \in A\} = 2 \cdot 3^{-2}.$$

It is easy to see that this process can be repeated as many times as desired to obtain as close an approximation as one wishes to g, by polynomials in functions from $\mathfrak{S}$. This completes the proof.*

The reader may find it instructive to try to carry over the proof of Proposition II_f to the compact infinite case, and to see just where it breaks down.

Part (C) of our thesis is amply supported by the Stone-Weierstrass theorem.

II_n. *Let B be any noncompact subset of R. Then there is a separating set $\mathfrak{S}$ of bounded continuous real-valued functions on B such that not every bounded continuous real-valued function on B is the uniform limit of polynomials in functions from $\mathfrak{S}$.*

Proof. The function arctan (x) (principal value) with domain restricted to B by itself separates the points of B and will be taken as constituting the set $\mathfrak{S}$. Suppose that B is unbounded above. Then there is a subset of B of the form $\{x_1, x_2, x_3, \cdots, x_n, \cdots\}$ such that $x_1 < x_2 < x_3 < \cdots < x_n < \cdots$ and $\lim_{n\to\infty} x_n = \infty$. Let $f(x_n) = (-1)^n$ $(n = 1, 2, 3, \cdots)$, let f be linear in the intervals

---

* Proposition II_e may fail for complex-valued continuous functions defined on a compact subset A of R. This is a fairly recondite fact. W. Rudin has shown [17] that if A is a compact subset of R that contains a "replica" of Cantor's ternary set, then there is a separating set of continuous complex-valued functions $\mathfrak{S}$ on A such that not every complex-valued continuous function on A is the uniform limit of polynomials (with complex coefficients) formed from the functions of $\mathfrak{S}$. It is not hard to see that if one allows the formation of complex conjugates as well as polynomials, then Proposition II_e holds for complex-valued continuous functions and polynomials with complex coefficients. Curiously enough, Proposition II_e holds without any exception for *quaternion*-valued continuous functions, as has been shown by J. C. Holladay [10]. The theorem follows immediately from the identity $q - iqi - jqj - kqk = 4a$, where q is the quaternion $a + bi + cj + dk$, i, j, and k are the usual quaternion units, and a, b, c, and d are real numbers. Holladay's observation has recently been generalized by Alvin Hausner [A generalized Stone-Weierstrass theorem, Arch. der Math., vol. 10, 1959, pp. 85–87] to the case of continuous functions defined on compact sets and having values in a Cayley-Dickson algebra of dimension 2^n $(n = 2, 3, \cdots)$ or a Clifford algebra of dimension 2^{2l} $(l = 1, 2, 3, \cdots)$. Hausner's proof generalizes Holladay's.

$]x_n, x_{n+1}[$ $(n=1, 2, 3, \cdots)$, and $f(x)=-1$ for $x<x_1$. Then f is a bounded continuous real-valued function on R and hence, with its domain restricted to B, is continuous on B. Plainly $\lim_{n\to\infty} f(x_n)$ does not exist. However, if φ is a function on B that is the uniform limit of polynomials in arctan (x), then $\lim_{n\to\infty} \varphi(x_n)$ must exist. Therefore f cannot be the uniform limit of polynomials in arctan (x).

The cases in which B is bounded above but unbounded below, or bounded but nonclosed, are treated very similarly. We omit the details.

Example III. The third illustration of our thesis is drawn from functional analysis. It deals with integral representations for certain linear functionals.

III_f. *Let F be a finite set (a subset of R or not) and L a real-valued functional defined for all real-valued functions on F such that: $L(\alpha f+\beta g)=\alpha L(f)+\beta L(g)$ for all functions f and g on F and all real numbers α and β (that is, L is linear); $L(f)\geqq 0$ if $f(x)\geqq 0$ for all $x\in F$ (that is, L is a positive linear functional). Then there is a nonnegative real-valued function λ defined on F such that $L(f)=\sum_{x\in F} f(x)\lambda(x)$ for all real-valued functions f on F.*

Like Proposition II_f, Proposition III_f has a somewhat artificial air. Nevertheless, it has great importance in linear algebra (see for example the discussion in [6], pp. 20–24).

Proposition III_f is grotesquely easy to prove. Write F as $\{x_1, x_2, \cdots, x_p\}$, where the points $x_1, x_2, \cdots, x_p$ are all distinct. Let φ_j be the function on F such that $\varphi_j(x_j)=1$ and

$$\varphi_j(x_k) = 0 \text{ for } j \neq k \ (j = 1, 2, \cdots, p; k = 1, 2, \cdots, j-1, j+1, \cdots, p).$$

Then any real-valued function f on F can be written in just one way in the form $f=\sum_{j=1}^{p} f(x_j)\varphi_j$. If L is a linear functional defined for all real-valued functions on F, then we have $L(f)=L(\sum_{j=1}^{p} f(x_j)\varphi_j)=\sum_{j=1}^{p} f(x_j)L(\varphi_j)$. Since φ_j is a nonnegative function, we have $L(\varphi_j)\geqq 0$ if L is a positive linear functional. Hence, writing $\lambda_j=L(\varphi_j)$ $(j=1, 2, \cdots, p)$, we obtain Proposition III_f.

The reader may well have noticed that positivity of L is really unnecessary in Proposition III_f: if L is *any* real-valued linear functional on all real-valued functions on F, then $L(f)=\sum_{x\in F} f(x)\lambda(x)$, where λ is now a real-valued function on F that may assume values of arbitrary sign. Here, as in comparing the proofs of Propositions II_f and II_c, we encounter a genuine difference between the finite and the infinite compact cases. In order to obtain an analogue of Proposition III_f valid for all compact subsets of R, we are forced to restrict L sharply. As usual, we replace the set of *all* real-valued functions by the set of *continuous* real-valued functions. The following theorem is known as F. Riesz's representation theorem.

III_c. *Let A be any compact subset of R. Let L be any real-valued functional defined for all continuous real-valued functions on A such that: $L(\alpha f+\beta g)=\alpha L(f)+\beta L(g)$ for all real numbers α and β and continuous real-valued functions f and g on A (L is linear); $L(f)\geqq 0$ if f is a nonnegative real-valued continuous function*

on A. Then there is a measure λ on the family $\mathfrak{B}(A)$ of Borel subsets of A such that $L(f)=\int_A f(x)d\lambda(x)$ for all continuous real-valued functions f on A.*†

A few words of explanation of Proposition III_c may be appropriate. A subset E of A is closed if it is a closed subset of R, *i.e.*, if it contains all of its own limit points. Let $\mathfrak{F}(A)$ denote the family of all closed subsets of A. There are families of subsets $\mathcal{G}$ of A that contain $\mathfrak{F}(A)$ and are also closed under the formation of countable unions and complements relative to A: that is, if $Y_1, Y_2, Y_3, \cdots, Y_n, \cdots$ are in $\mathcal{G}$, then $\bigcup_{n=1}^{\infty} Y_n$ is in $\mathcal{G}$, and if Y is in $\mathcal{G}$, then $Y' \cap A$ is in $\mathcal{G}$. The family of all subsets of A is an example. From this it is easy to see that there is a smallest family, $\mathfrak{B}(A)$, of subsets of A, that contains $\mathfrak{F}(A)$ and is closed under the formation of countable unions and of complements relative to A. The family $\mathfrak{B}(A)$ is called the family of Borel subsets of A. A real-valued function λ defined for every set in $\mathfrak{B}(A)$ is called a measure (in this essay) if $0 \leqq \lambda(E) < \infty$ for all E in $\mathfrak{B}(A)$ and $\lambda(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \lambda(E_n)$ for every countably infinite subfamily $\{E_1, E_2, E_3, \cdots, E_n, \cdots\}$ of $\mathfrak{B}(A)$ for which $E_n \cap E_m = \varnothing$ if $n \neq m$ ($\varnothing$ denotes the void set). This property of λ is called countable additivity.

The most remarkable feature of Proposition III_c is the fact that the linear functional L has a representation as an integral with respect to a *countably* additive measure. Countable additivity is the property of measures that makes the most powerful theorems of integration theory valid (*e.g.*, Lebesgue's theorem on term-by-term integration of convergent sequences of functions, and Fubini's theorem); and it is a quite delicate property. As we shall see below, countable additivity of the measure that gives the integral representation for L depends entirely upon the compactness of A.

We proceed to the proof of Proposition III_c. There are a number of measure-theoretic technicalities in it, a complete explanation of which would require several pages. We shall therefore limit ourselves to a sketch of the construction, giving references to the literature where needed.

We build up the measure λ from the functional L in steps. First, let F be any closed subset of A. Define $\lambda(F)$ as the greatest lower bound of the set $\{L(\varphi)\colon \varphi$ is a continuous real-valued function on A, $\varphi(x) \geqq 1$ for $x \in F$ and $\varphi(x) \geqq 0$ for all $x \in A\}$. It is clear that $0 \leqq \lambda(F) \leqq L(1) < \infty$, that $\lambda(A) = L(1)$, and that $\lambda(\varnothing) = 0$. Next, suppose that G is an open subset of A (*i.e.*, that $G' \cap A$ is closed).‡ Then define $\lambda'(G)$ as the least upper bound of the set of numbers $\{\lambda(F)\colon F$ is closed in A and $F \subset G\}$. Again it is clear that $0 \leqq \lambda'(G) \leqq L(1) < \infty$. If a subset

* Here and below all integrals are of Lebesgue's type, unless the contrary is explicitly stated.

† The theorem was proved by F. Riesz [16] for the case in which $A = [0, 1]$, and with the integral $\int_A f(x)d\lambda(x)$ replaced by a Riemann-Stieltjes integral $\int_0^1 f(x)dl(x)$, where l is a function of finite variation on $[0, 1]$. For a general compact subset A of R, it is most convenient to present the theorem as we have done, in terms of measures and integrals of Lebesgue's type.

‡ Open subsets of A need not be open when considered as subsets of R. For example, if $A = \{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\}$, then any finite set excluding 0 is open in A. An open subset of A, however, is always the intersection with A of a countable union of intervals $]a, b[$. Closed subsets of A, on the other hand, are always closed in R, since A itself is closed in R.

G of A is both open and closed, then $\lambda(G)=\lambda'(G)$, since $\lambda(F_1)\leqq\lambda(F_2)$ whenever F_1 and F_2 are closed in A and $F_1\subset F_2$. Finally, for an arbitrary subset X of A, let $\lambda''(X)$ be the greatest lower bound of the set of numbers $\{\lambda'(G): G \text{ is open in } A \text{ and } G\supset X\}$. It is obvious that $\lambda'(G)=\lambda''(G)$ if G is open in A and that $\lambda''(F)\geqq\lambda(F)$ if F is closed in A. To prove that $\lambda''(F)\leqq\lambda(F)$ for F closed in A, let ϵ and α be any positive real numbers, and select a continuous real-valued function φ on A such that $\varphi(x)\geqq 1$ for $x\in F$, $\varphi(x)\geqq 0$ for $x\in A$, and $\lambda(F)>L(\varphi)-\epsilon$. Let $\psi=(1+\alpha)\varphi$, $F_1=\{x: x\in A,\ \psi(x)\geqq 1\}$, and $G_1=\{x: x\in A,\ \psi(x)>1\}$. The definitions of λ and λ' show that $\lambda(F_1)\geqq\lambda'(G_1)$. Now we have $L(\varphi)=(1/(1+\alpha))L(\psi)$, and $L(\psi)\geqq\lambda(F_1)\geqq\lambda'(G_1)$. Since $G_1\supset F$, we also have $\lambda'(G_1)\geqq\lambda''(F)$. Combining these estimates, we have

$$L(F) > L(\varphi) - \epsilon = \frac{1}{1+\alpha} L(\psi) - \epsilon \geqq \frac{1}{1+\alpha}\lambda'(G_1) - \epsilon \geqq \frac{1}{1+\alpha}\lambda''(F) - \epsilon.$$

Since α and ϵ can be arbitrarily small, it follows that $\lambda(F)\geqq\lambda''(F)$. Hence $\lambda=\lambda''$ whenever λ is defined, and we may regard λ'' as an extension of λ. For typographical convenience, we will now drop the "$''$" and write $\lambda(X)$ instead of $\lambda''(X)$, for all subsets X of A.

We next show that λ is a Carathéodory outer measure (see for example [18], pp. 43–47). That is,

$$\lambda(X) \leqq \lambda(Y) \qquad \text{for } X \subset Y \subset A, \tag{1}$$

$$\lambda\left(\bigcup_{n=1}^{\infty} X_n\right) \leqq \sum_{n=1}^{\infty} \lambda(X_n) \qquad \text{if } X_n \subset A \qquad (n = 1, 2, 3, \cdots). \tag{2}$$

The inequality (1) is obviously valid. Inequality (2) is at the root of countable additivity of λ for Borel sets and, not surprisingly, depends upon the compactness of A.

The weaker inequality

$$\lambda(X_1 \cup X_2 \cup \cdots \cup X_p) \leqq \sum_{n=1}^{p} \lambda(X_n), \qquad \bigcup_{n=1}^{p} X_n \subset A, \tag{2'}$$

is established by a routine argument from the foregoing definitions: we will not reproduce it. (See for example [11], p. 1009, footnote 12.) Inequality (2') does *not* depend upon compactness of A. We prove (2) from (2'). If the series on the right side of (2) diverges, there is nothing to prove. Suppose that this series converges, and let ϵ be any positive real number. For $n=1, 2, 3, \cdots$, let G_n be an open subset of A such that $G_n\supset X_n$ and $\lambda(G_n)-\epsilon 2^{-n}<\lambda(X_n)$. Let F be a closed subset of A such that $F\subset\bigcup_{n=1}^{\infty} G_n$ and $\lambda(F)+\frac{1}{2}\epsilon>\lambda(\bigcup_{n=1}^{\infty} G_n)$. As a closed subset of A, F is compact, and each G_n is the intersection with A of a countable union of open intervals: $G_n=(\bigcup_{k=1}^{\infty} I_{n,k})\cap A$, where each $I_{n,k}$ is an open interval. Thus we have $F\subset\bigcup_{n=1}^{\infty}(\bigcup_{k=1}^{\infty} I_{n,k})$. By property (3) of compact sets, we have $F\subset I_{n_1,k_1}\cup\cdots\cup I_{n_l,k_l}$, and hence $F\subset(\bigcup_{k=1}^{\infty} I_{n_1,k})\cup\cdots\cup(\bigcup_{k=1}^{\infty} I_{n_l,k})$. This

implies that $F \subset G_{n_1} \cup \cdots \cup G_{n_l}$. Properties (1) and (2′) of λ imply that $\lambda(F) \leqq \lambda(G_{n_1} \cup \cdots \cup G_{n_l}) \leqq \lambda(G_{n_1}) + \cdots + \lambda(G_{n_l})$. We can now put together our estimates to find

$$\lambda\left(\bigcup_{n=1}^{\infty} X_n\right) \leqq \lambda\left(\bigcup_{n=1}^{\infty} G_n\right) < \lambda(F) + \tfrac{1}{2}\epsilon \leqq \lambda(G_{n_1}) + \cdots + \lambda(G_{n_l}) + \tfrac{1}{2}\epsilon$$

$$\leqq \sum_{n=1}^{\infty} \lambda(G_n) + \tfrac{1}{2}\epsilon < \sum_{n=1}^{\infty} (\lambda(X_n) + \epsilon 2^{-n}) + \tfrac{1}{2}\epsilon = \sum_{n=1}^{\infty} \lambda(X_n) + \epsilon.$$

Since ϵ can be chosen arbitrarily small, inequality (2) is demonstrated.

The argument now follows a familiar course. A set $X \subset A$ is (Carathéodory) measurable if $\lambda(Y) = \lambda(Y \cap X) + \lambda(Y \cap X')$ for all $Y \subset A$. The family $\mathfrak{M}$ of measurable sets is proved to be closed under the formation of countable unions and complements relative to A, and λ is proved countably additive on $\mathfrak{M}$. This can be taken *verbatim* from [18], pp. 44–46. One next shows that closed subsets of A are measurable. This again is done by a routine argument, which we omit.

To finish the proof, we must show that

$$L(f) = \int_A f(x)\, d\lambda(x) \tag{3}$$

for all continuous real-valued functions f on A. The equality (3) is obvious if f is a constant function. If f is nonconstant, let $c = \min\{f(x): x \in A\}$, $d = \max\{f(x): x \in A\}$, and let n be any positive integer. Let $\alpha_k = [(n-k)c + kd]/n$ for $k = 0, 1, 2, \cdots, n$. That is, the numbers $c = \alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n = d$ cut up the interval $[c, d]$ into n subintervals of equal length. We now subdivide A as is often done in constructing Lebesgue integrals: let $A = D_0 = \{x: x \in A, f(x) \geqq \alpha_0\}$, $D_1 = \{x: x \in A, f(x) \geqq \alpha_1\}$, $\cdots$, $D_k = \{x: x \in A, f(x) \geqq \alpha_k\}$, for $k = 0, 1, 2, 3, \cdots, n$. We also write E_k for the set $\{x: x \in A, \alpha_k \leqq f(x) < \alpha_{k+1}\}$ for $k = 0, 1, 2, \cdots, n-1$, and $E_n = \{x: x \in A, f(x) = \alpha_n\} = D_n$. Then plainly enough $E_k = D_k \cap D_{k+1}'$ for $k = 0, 1, \cdots, n-1$.

We now write the function f as a linear combination of other continuous functions, in terms of which we can estimate the numbers $\lambda(E_0), \lambda(E_1), \cdots, \lambda(E_n)$. For $k = 1, 2, \cdots, n$, let f_k be the function on A such that

$$f_k(x) = 0 \qquad \text{if } f(x) \leqq \alpha_{k-1},$$

$$f_k(x) = \frac{1}{\alpha_k - \alpha_{k-1}} (f(x) - \alpha_{k-1}) \qquad \text{if } \alpha_{k-1} < f(x) < \alpha_k,$$

$$f_k(x) = 1 \qquad \text{if } f(x) \geqq \alpha_k.$$

It is clear that f_k is a continuous real-valued function on A, and it follows from the definition of λ that $L(f_k) \geqq \lambda(D_k)$ $(k = 1, 2, \cdots, n)$. Finally, it is easy to check that

$$f = \alpha_0 + (\alpha_1 - \alpha_0) f_1 + (\alpha_2 - \alpha_1) f_2 + \cdots + (\alpha_n - \alpha_{n-1}) f_n.$$

Now apply the functional L to both sides of this identity, and use the inequalities $L(f_k) \geqq \lambda(D_k)$. This yields

$$\begin{aligned} L(f) &= \alpha_0 L(1) + (\alpha_1 - \alpha_0)L(f_1) + (\alpha_2 - \alpha_1)L(f_2) + \cdots + (\alpha_n - \alpha_{n-1})L(f_n) \\ &\geqq \alpha_0\lambda(A) + (\alpha_1 - \alpha_0)\lambda(D_1) + (\alpha_2 - \alpha_1)\lambda(D_2) + \cdots + (\alpha_n - \alpha_{n-1})\lambda(D_n) \\ &= \alpha_0(\lambda(D_0) - \lambda(D_1)) + \alpha_1(\lambda(D_1) - \lambda(D_2)) + \alpha_2(\lambda(D_2) - \lambda(D_3)) + \cdots \\ &\quad + \alpha_{n-1}(\lambda(D_{n-1}) - \lambda(D_n)) + \alpha_n\lambda(D_n). \end{aligned}$$

We also have $\lambda(D_k) = \lambda(D_{k+1}) + \lambda(E_k)$ for $k = 0, 1, \cdots, n-1$, since λ is a (countably) additive measure. Thus

$$L(f) \geqq \alpha_0\lambda(E_0) + \alpha_1\lambda(E_1) + \cdots + \alpha_{n-1}\lambda(E_{n-1}) + \alpha_n\lambda(E_n).$$

The right side of the last inequality is the integral $\int_A h_n(x)d\lambda(x)$, where $h_n(x) = \alpha_k$ for $x \in E_k$ $(k = 0, 1, \cdots, n)$. Since $\lim_{n\to\infty} h_n(x) = f(x)$ for every $x \in A$, a standard theorem on convergence of Lebesgue integrals implies that $\lim_{n\to\infty} \int_A h_n(x)d\lambda(x) = \int_A f(x)d\lambda(x)$. (See for example [18], p. 29, Theorem (12.11).) It follows that $L(f) \geqq \int_A f(x)d\lambda(x)$. Replacing f by $-f$, we have $-L(f) = L(-f) \geqq \int_A -f(x)d\lambda(x) = -\int_A f(x)d\lambda(x)$, which shows that $L(f) \leqq \int_A f(x)d\lambda(x)$. Thus $L(f) = \int_A f(x)d\lambda(x)$ for every continuous real-valued function on A. This completes the proof.*

It should be remarked that Proposition III_c admits an immediate generalization. Suppose that L is a linear functional defined for all continuous real-valued functions on A that is not positive but satisfies the weaker condition $|L(f)| \leqq C \cdot \max\{|f(x)| : x \in A\}$ for all f, where C is a constant independent of f. Then one can prove that $L = L_1 - L_2$, where L_1 and L_2 are linear and positive (see for example [3], p. 245, Theorem 8). Thus $L(f) = \int_A f(x)d\lambda(x)$, where $\lambda = \lambda_1 - \lambda_2$ and λ_1 and λ_2 are measures on $\mathfrak{B}(A)$.

Part (C) of our thesis is fully supported by the Riesz representation theorem.

III_n. *Let B be any noncompact subset of R. Then there is a positive linear functional L defined for all bounded continuous real-valued functions on B that is representable by no countably additive measure defined on the Borel sets of B.*

Proof. Suppose that B is unbounded above and that $\{x_n\}_{n=1}^{\infty}$ is a subset of B such that $x_1 < x_2 < x_3 < \cdots < x_n < \cdots$ and $\lim_{n\to\infty} x_n = \infty$. For every bounded continuous real-valued function f on B, let $p(f) = \overline{\lim}_{n\to\infty} f(x_n)$. We then have $p(f+g) \leqq p(f) + p(g)$ and $p(\alpha f) = \alpha p(f)$ if f and g are bounded continuous real-valued functions on B and α is a nonnegative real number. The Hahn-Banach theorem implies (see [2], p. 29, Corollaire) that there is a linear functional L defined for all bounded continuous real-valued functions f on B such that $L(f) \leqq p(f)$ for all f.† Thus we have $-p(-f) \leqq L(f) \leqq p(f)$. Since $-p(-f) = \underline{\lim}_{n\to\infty} f(x_n)$, we may write

* The proof just given is based on a proof given by Kakutani ([11], Theorem 9). The history of representation theorems like Proposition III_c is long and involved, and we cannot go into it here.

† This is a sophisticated result, which depends upon the well-ordering theorem.

$$\underline{\lim}_{n\to\infty} f(x_n) \leqq L(f) \leqq \overline{\lim}_{n\to\infty} f(x_n).$$

Thus, if $f(x)\geqq 0$ for all $x\in B$, we have $\underline{\lim}_{n\to\infty} f(x_n)\geqq 0$ and thus $L(f)\geqq 0$. Also, if $\lim_{n\to\infty} f(x_n)=0$, it follows that $L(f)=0$. Now suppose that L can be represented by an integral with respect to a measure λ defined at least for the closed subsets E of B. Let f be any continuous function on B such that $0\leqq f(x)\leqq 1$ for all $x\in B$ and $f(x)\geqq 1$ for $x\in E$. A moment's reflection shows that $L(f)\geqq\lambda(E)$. However, if E is bounded, f can be chosen so that $f(x_n)=0$ for all sufficiently large n. That is, we must have $\lambda(E)=0$ for all closed bounded subsets E of B. The set B is the union of an increasing sequence of such sets: $B=\bigcup_{k=1}^{\infty} E_k$. If λ were countably additive, we would have $\lambda(B)=\lim_{k\to\infty}\lambda(E_k)=0$. But $\lambda(B)=L(1)=1$. Thus there is no countably additive measure defined for closed subsets of B that produces an integral representation for L. A virtually identical argument is used if B is bounded and nonclosed.

Remark. Something can be salvaged from Riesz's representation theorem for positive linear functionals on the bounded continuous real-valued functions on noncompact sets. Such functionals have integral representations, but with respect to *finitely* and not always *countably* additive measures. The main theorems of integration theory (term-by-term integration of sequences, and Fubini's theorem) are in general false for finitely additive measures, so that the analytic utility of such representations is small. The properties of, and representations using, finitely additive measures are discussed at length in [22] and [8].

Example IV. At the risk of tediousness, we cite still another example in support of our thesis. Consider the set of *all* real-valued continuous functions on a subset A of R, denoted by the symbol $\mathfrak{C}(A)$. The set $\mathfrak{C}(A)$ is a ring under the operations of pointwise addition and multiplication. As such, $\mathfrak{C}(A)$ admits the concept of an *ideal*: an ideal $\mathfrak{I}$ in $\mathfrak{C}(A)$ is a nonvoid subset of $\mathfrak{C}(A)$ such that if f, g are in $\mathfrak{I}$ and φ is in $\mathfrak{C}(A)$, then $f-g$ is in $\mathfrak{I}$ and φf is in $\mathfrak{I}$. To avoid uninteresting special cases in our proofs, we suppose that all ideals considered are distinct from $\mathfrak{C}(A)$ itself.

It is simple to find all ideals in $\mathfrak{C}(F)$ if F is a finite subset of R.

IV$_f$. *Let F be a finite subset of R, so that $\mathfrak{C}(F)$ is the ring of all real-valued functions on F. Let $\mathfrak{I}$ be an ideal in $\mathfrak{C}(F)$. Then there is a nonvoid subset H of F such that $\mathfrak{I}$ consists exactly of the functions in $\mathfrak{C}(F)$ that vanish on H.*

Proof. If $\mathfrak{I}$ consists of the function 0 and nothing else, the result asserted holds with $H=F$. Suppose then that $\mathfrak{I}\neq\{0\}$, that $f\in\mathfrak{I}$, and $f(x_j)\neq 0$, where as before we write $F=\{x_1, x_2, \cdots, x_p\}$. Defining φ_j as in the proof of Proposition III$_f$, we see that $(1/f(x_j))\varphi_j f=\varphi_j\in\mathfrak{I}$. Let $J=\{x_{j_1}, x_{j_2}, \cdots, x_{j_l}\}$ be the set of all $x_j\in F$ for which there is an $f\in\mathfrak{I}$ such that $f(x_j)\neq 0$. Then every real-valued function g on F that vanishes on $H=F\cap J'$ has the form $g=g(x_{j_1})\varphi_{j_1}+\cdots+g(x_{j_l})\varphi_{j_l}$ and is hence in $\mathfrak{I}$. Clearly every function in $\mathfrak{I}$ must vanish on H, and thus IV$_f$ is proved.

The infinite compact analogue of Proposition IV_f, if stated directly, is rather weak: we cannot classify *all* ideals in $\mathfrak{C}(A)$ for any infinite subset A of R. However, all ideals in $\mathfrak{C}(A)$ have a very special property if A is compact.

IV_c. *Let A be a compact subset of R, and let $\mathfrak{I}$ be an ideal in $\mathfrak{C}(A)$. Then there is at least one point $p\in A$ such that $f(p)=0$ for all $f\in\mathfrak{I}$.*

Proof. Assume that $\mathfrak{I}$ is an ideal in $\mathfrak{C}(A)$ and that for every $p\in A$, there is some $f_p\in\mathfrak{I}$ such that $f_p(p)\neq 0$. Then $f_p^2\in\mathfrak{I}$ and $f_p^2(p)>0$. Since f_p^2 is continuous, there is an open interval I_p containing p such that $f_p^2(t)>0$ for all $t\in A\cap I_p$. Since A is compact, a finite number of the intervals I_p, say $I_{p_1}, I_{p_2}, \cdots, I_{p_m}$, cover A: A is contained in $I_{p_1}\cup I_{p_2}\cup\cdots\cup I_{p_m}$. If we examine the function $g=f_{p_1}^2+\cdots+f_{p_m}^2$, we see that $g\in\mathfrak{I}$ and $g(x)>0$ for all $x\in A$. Then, for every $\varphi\in\mathfrak{C}(A)$, we have $\varphi g^{-1}g\in\mathfrak{I}$, so that $\mathfrak{I}$ is not an ideal in our present sense.

Proposition IV_c has the following immediate corollary. *Let $\mathfrak{M}$ be a maximal ideal in $\mathfrak{C}(A)$, where A is any compact subset of R. (An ideal $\mathfrak{M}$ in $\mathfrak{C}(A)$ is said to be maximal if there is no ideal $\mathfrak{N}$ in $\mathfrak{C}(A)$ such that $\mathfrak{M}\subsetneq\mathfrak{N}\subsetneq\mathfrak{C}(A)$.) Then there is a point $p\in A$ such that $\mathfrak{M}$ consists of all functions in $\mathfrak{C}(A)$ that vanish at p.* Thus the maximal ideals of $\mathfrak{C}(A)$ are readily classified.

To see the complications that arise if one attempts to find all of the ideals in $\mathfrak{C}(A)$, consider $\mathfrak{C}([0, 1])$ and the set of all functions $f\in\mathfrak{C}([0, 1])$ that vanish in some interval $[0, \delta]$, where $0<\delta\leqq 1$ (δ depends upon f). Clearly this set is an ideal in $\mathfrak{C}([0, 1])$, and it is contained in only one maximal ideal. A whole class of ideals is found by the following construction. Choose a sequence $\{t_n\}_{n=1}^{\infty}$ of numbers such that $0<t_n\leqq 1$ for all n and $\lim_{n\to\infty} t_n=0$. Consider the set of all $f\in\mathfrak{C}([0, 1])$ such that $f(t_n)=0$ for $n\geqq N$ (N depends upon f). Finally, consider all $f\in\mathfrak{C}([0, 1])$ for which $|x^{-1}f(x)|$ is bounded for $0<x\leqq 1$. Here again we have an ideal contained in only one maximal ideal. These examples give a hint of the complexity of the class of all ideals in $\mathfrak{C}([0, 1])$. Similar constructions can be given in $\mathfrak{C}(A)$ for any infinite compact subset A of R.

Nevertheless, Proposition IV_c can be greatly improved on. The reader will see that the following assertion bears much the same relation to IV_f as II_c bears to II_f.

IV_c'. *Let A be a compact subset of R and $\mathfrak{I}$ an ideal in $\mathfrak{C}(A)$ that is closed under the formation of uniform limits. That is, if $f_1, f_2, f_3, \cdots, f_n, \cdots$ are in $\mathfrak{I}$, and $\lim_{n\to\infty} f_n=\varphi$, the limit being uniform on A, then φ is also in $\mathfrak{I}$. In this case, there is a nonvoid closed subset E of A such that $\mathfrak{I}$ consists exactly of the functions in $\mathfrak{C}(A)$ that vanish on E. Conversely, if E is any nonvoid closed subset of A, the set of all functions in $\mathfrak{C}(A)$ that vanish on E is an ideal in $\mathfrak{C}(A)$ closed under the formation of uniform limits.*

Proof. The second assertion is very easy to prove: we shall not go into its details. To prove the first assertion, let E be the set of all $x\in A$ such that $f(x)=0$ for all $f\in\mathfrak{I}$. Proposition IV_c shows that E is nonvoid. It is quite obvious that

every function in $\mathfrak{J}$ vanishes everywhere on the set E. We must prove that every function φ in $\mathfrak{C}(A)$ that vanishes on E is the uniform limit (on A) of a sequence of functions $\{f_n\}_{n=1}^{\infty}$ in $\mathfrak{J}$: this will show that φ is in the ideal $\mathfrak{J}$. If $E=A$, there is nothing to prove: in this case, $\mathfrak{J}=\{0\}$. Suppose that $E\subsetneq A$ and that D is any closed subset of A disjoint from E. Let p be an arbitrary point of D. Since p is not in E, there is a function $f_p\in\mathfrak{J}$ such that $f_p(p)\neq 0$. The functions f_p^2 are nonnegative and lie in $\mathfrak{J}$. We argue as in the proof of Proposition IV$_c$ to infer that there are a finite number of points $p_1, p_2, \cdots, p_m$ in D such that the function $f=f_{p_1}^2+f_{p_2}^2+\cdots+f_{p_m}^2$ is positive throughout D. Since D is compact, f has a positive lower bound, say α, on D. Since f is in $\mathfrak{J}$, f vanishes throughout E. Define the function g by the relations

$$g(x)=\frac{\alpha}{f(x)} \text{ if } f(x)\geqq\alpha, \qquad g(x)=1 \text{ if } f(x)<\alpha.$$

Plainly g is continuous on A, and the function $h=(1/\alpha)gf$ is in $\mathfrak{J}$. Also we have $h(x)=1$ for $x\in D$, $h(x)=0$ for $x\in E$, and $0\leqq h(x)\leqq 1$ for all $x\in A$.

Now let φ be any function in $\mathfrak{C}(A)$ such that $\varphi(x)=0$ for all $x\in E$. Let $D_n=\{x: x\in A, |\varphi(x)|\geqq 1/n\}$ $(n=1, 2, 3, \cdots)$. If $\varphi\neq 0$, then D_n is nonvoid for all sufficiently large integers n. For all such n, let h_n be a function in $\mathfrak{J}$ such that $h_n(x)=1$ for $x\in D_n$ and $0\leqq h_n(x)\leqq 1$ for all $x\in A$. Such an h_n exists because D_n is a closed subset of A disjoint from E. Since h_n is in $\mathfrak{J}$, the function $h_n\varphi$ is in $\mathfrak{J}$. The construction of h_n shows that $|h_n(x)\varphi(x)-\varphi(x)|\leqq 1/n$ for all $x\in A$. Thus $\lim_{n\to\infty} h_n\varphi=\varphi$, the limit being uniform on A. This completes the proof.

Proposition IV fails in a most satisfactory way for noncompact subsets of R.

IV$_n$. *Let B be any noncompact subset of R. Then there is an ideal $\mathfrak{J}$ in $\mathfrak{C}(B)$ closed under the formation of uniform limits such that for every $x\in B$, there is an $f\in\mathfrak{J}$ such that $f(x)\neq 0$.*

Proof. Suppose that B is unbounded above. For every positive integer n, let f_n be the function on R such that $f_n(x)=0$ for $x\geqq n$, $f_n(x)=1$ for $x\leqq n-1$, and f_n is linear in the interval $[n-1, n]$. With its domain restricted to B, f_n is clearly in $\mathfrak{C}(B)$. Let $\mathfrak{J}_0$ be the set of all functions of the form $\varphi=\varphi_1 f_{n_1}+\varphi_2 f_{n_2}+\cdots+\varphi_l f_{n_l}$, where $\varphi_1, \varphi_2, \cdots, \varphi_l$ are in $\mathfrak{C}(B)$. Obviously $\mathfrak{J}_0$ is an ideal in $\mathfrak{C}(B)$ (in fact the smallest ideal containing $f_1, f_2, f_3, \cdots$). It is clear that $\mathfrak{J}_0$ is not all of $\mathfrak{C}(B)$, since $\varphi(x)=0$ for $x\geqq\max\{n_1, n_2, \cdots, n_l\}$ and B is unbounded above. Since all f_n are in $\mathfrak{J}_0$ and $f_n(x)>0$ for $n>[x]+1$, it is clear that $\mathfrak{J}_0$ has the property asserted. Let $\mathfrak{J}$ be the set of all uniform limits of sequences of functions in $\mathfrak{J}_0$. Then $\mathfrak{J}$ is an ideal, as can be easily verified, and $\mathfrak{J}$ is not equal to $\mathfrak{C}(B)$ since $\lim_{x\to\infty}\varphi(x)=0$ for all $\varphi\in\mathfrak{J}$. Very similar constructions are used if B is unbounded below or nonclosed.

Compactness in more general spaces. Suppose now that we have any metric space, say X. That is, for every pair of points (x, y) from X, a distance $\rho(x, y)$

is defined, satisfying the following axioms: $\rho(x, x)=0$, and $\rho(x, y)$ is a positive real number if $x \neq y$; $\rho(x, y)=\rho(y, x)$ for all x and y in X; $\rho(x, z) \leqq \rho(x, y)+\rho(y, z)$ for all x, y, and z in X. A sequence $\{t_n\}_{n=1}^{\infty}$ of elements of X converges to a point t of X if $\lim_{n \to \infty} \rho(t_n, t)=0$. A limit point of a subset Y of X is a point t such that for every positive real number ϵ, there is some y in Y such that $0<\rho(t, y)<\epsilon$. A subset G of X is open if for every $x \in G$, there is a positive real number α such that G contains all points $y \in X$ such that $\rho(x, y)<\alpha$.* With these concepts, compactness of the metric space X can be defined by repeating properties (1) and (2) of compactness *verbatim*. Property (3) of compactness is changed slightly, to the following.

(3′) Let $\{G_\iota\}_{\iota \in I}$ be a family of open subsets of X such that $\bigcup_{\iota \in I} G_\iota = X$. Then there is a finite subfamily $\{G_{\iota_1}, G_{\iota_2}, \cdots, G_{\iota_m}\}$ of $\{G_\iota\}_{\iota \in I}$ such that $G_{\iota_1} \cup G_{\iota_2} \cup \cdots \cup G_{\iota_m} = X$.

Properties (1), (2), and (3′) are equivalent for a metric space, and are collectively called compactness. Furthermore, all parts of all of the propositions I, II, III, and IV remain true for metric spaces. (A real-valued function f on a metric space X is continuous if $x_n \to x$ in X implies $f(x_n) \to f(x)$ in R: similarly with complex- or quaternion-valued functions on X.) The proofs of Propositions I_f, II_f, III_f, and IV_f are trivially unchanged, since a function on a finite set F is indifferent to the imbedding of F in the line, or in any other metric space for that matter. The proofs of Propositions I_c, II_c, III_c, and IV_c are repeated with the set A replaced by an arbitrary compact metric space. A little care is needed with Propositions I_n, II_n, III_n, and IV_n, where B is replaced by an arbitrary noncompact metric space. However, examples showing the falsity of the four propositions can be constructed. We shall not go into details here (see [9], p. 69, Theorem 30; [7], p. 423, Theorem 3; [9], p. 53, Theorem 7). Generalizations of I_n, II_n, and IV_n, respectively, are given in these three references. The writer knows of no explicit generalization in print of III_n for noncompact metric spaces. Such a generalization is easily obtained, however, along lines very like those of the construction in the proof of Proposition III_n.

Compactness for the most general topological spaces exhibits peculiarities not found in metric spaces. A set X is called a topological space if there is prescribed in it a family $\mathfrak{O}$ of subsets, called *open sets*, satisfying the following axioms: $\varnothing$ is in $\mathfrak{O}$; if $G_1, G_2, \cdots, G_n$ are in $\mathfrak{O}$, then $G_1 \cap G_2 \cap \cdots \cap G_n$ is in $\mathfrak{O}$; if $\{G_\iota\}_{\iota \in I}$ is contained in $\mathfrak{O}$, then $\bigcup_{\iota \in I} G_\iota$ is in $\mathfrak{O}$. A subset E of X is closed if its complement E' (relative to X) is open. To avoid technicalities of no present interest, we suppose that all topological spaces X discussed are completely regular. That is, given any two points x and y in X distinct from each other, there is an open set G in X containing x but not y; and given a closed set E and a point $x \in X \cap E'$, there is a continuous real-valued function f defined on X such that $f(x)=0$ and $f(y)=1$ for all $y \in E$. (A real-valued function f on X is continuous if $f^{-1}(]a, b[)$

* There is a very extensive theory of metric spaces. Perhaps the most complete treatise on the subject is [14].

is an open subset of X for all open intervals $]a, b[$ of R.)* For general topological spaces, we take (3′) as the definition of compactness: every open covering of X admits a finite subcovering. In this very general situation, (1) and (2) are no longer equivalent to (3′). There is an analogue of (1) dealing with generalizations of sequences, which is equivalent to (3′) (see [12], p. 136, Theorem 2). There is also a reformulation of (2) which is equivalent to (3′) (see [12], p. 163, Problem I).

Our propositions I_c, II_c, III_c, and IV_c are true for all compact topological spaces. In fact, all of the proofs carry over without change if A is replaced by an arbitrary compact space X, except for the proof in I_c that every real-valued continuous function on X attains its least upper and greatest lower bounds. For this, a nearly trivial substitute proof using open coverings can be supplied.

Some changes appear in the propositions about noncompact spaces. Proposition I_n is false for general topological spaces: there are noncompact completely regular spaces on which every continuous real-valued function is bounded. These spaces seem first to have been studied by the writer ([9], pp. 67–71). Propositions II_n and IV_n hold for all noncompact completely regular spaces. (See [9], p. 53, Theorem 7 and [7], p. 423, Theorem 3, respectively.) Proposition III_n seems not to have been studied explicitly for noncompact spaces. There are some curious measure-theoretic and topological technicalities here; suffice it to say that there are noncompact completely regular spaces X such that every positive linear functional on the space of all bounded continuous real-valued functions can be represented by the integral with respect to a countably additive measure.†

Our final illustration of the power of compactness as an hypothesis is one which we can only cite without proof. Consider a group G. A unitary representation of G is a homomorphism $x \to U(x)$ mapping G into the group of unitary operators on some Hilbert space $\mathfrak{H}$. If there is no proper closed subspace of $\mathfrak{H}$ carried into itself by all of the operators $U(x)$, the representation U is called irreducible. It is a classical algebraic fact (which we state in somewhat disguised form) that if G is finite, then all of its irreducible representations are finite-dimensional. Also, every complex-valued function on G is a linear combination (with complex coefficients) of functions $(U(x)\xi, \eta)$, where U is an irreducible unitary representation of G and ξ, η are in the finite-dimensional complex Euclidean space $\mathfrak{H}$ on which U acts. This is merely a restatement of the classical fact that the group algebra of G over the complex numbers is semisimple.‡

For infinite compact groups,§ the Peter-Weyl theorem replaces the foregoing

---

* There is a highly developed theory of topological spaces. See for example the treatise [12].

† The space of ordinal numbers less than Ω with the interval topology is an example. The $\{0, 1\}$ measure that represents the functional $\lim_{\alpha \to \Omega} f(\alpha)$ is, however, not a *regular* measure.

‡ For a discussion of this theorem, see for example [4], p. 62, Satz 4.3.

§ A topological group is a group and a topological space in which the mapping $(x, y) \to xy^{-1}$ is continuous. A (locally) compact group is a topological group which as a topological space is (locally) compact.

result. Every irreducible continuous unitary representation of a compact group G is finite-dimensional. Every complex-valued continuous function on G can be arbitrarily uniformly approximated by linear combinations of functions $(U(x)\xi, \eta)$ where U is a continuous irreducible unitary representation of G and ξ, η are elements of the finite-dimensional complex Euclidean space $\mathfrak{H}$ on which U acts.*

If G is a noncompact, non-Abelian, locally compact group, then it may have no nontrivial finite-dimensional unitary representations. The group of all complex matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $ad-bc=1$ is an example.†

Other applications of compactness. The preceding sections of this essay treat compactness as an essential hypothesis in theorems about various entities met with in analysis. These theorems clearly show the vital part played by compactness in obtaining usable theorems. Perhaps as important as this, however, is the rôle played by compactness in existence theorems. The general idea used is the following. Suppose that one has a collection S of objects of a certain sort—they may be functions, measures, linear functionals, *etc.*—and suppose that one wishes to show the existence of one of the objects, s_0, having a certain property P. The property P is often numerical in character. Suppose further that S is topologized so as to be compact, and for simplicity let us say that S is a metric space. Suppose that one can find a sequence $\{s_1, s_2, s_3, \cdots, s_n, \cdots\}$ of objects in S that come closer and closer to having the property P. Then, extracting from this sequence a subsequence $\{s_{n_1}, s_{n_2}, s_{n_3}, \cdots, s_{n_k}, \cdots\}$ with limit s_0 in S, one can often show that s_0 actually has the property P. The argument is a little more sophisticated if S is compact but not metric, but the basic idea is identical.

There is a formidable list of existence theorems in analysis that employ this compactness technique. However, as Kipling said, that is another story.

References

1. Tom M. Apostol, Mathematical Analysis, Reading, Massachusetts, 1957.

2. Stefan Banach, Théorie des Opérations Linéaires, Monografje Matematyczne, vol. I, Warszawa, 1932.

3. Garrett Birkhoff, Lattice Theory, rev. ed., Amer. Math. Soc. Colloquium Publications, Vol. XXV, New York, 1948.

4. Hermann Boerner, Darstellungen von Gruppen, Grundlehren der Math. Wiss., Bd. LXXIV, Berlin-Göttingen-Heidelberg, 1955.

5. I. M. Gel'fand and M. A. Naĭmark, Unitary representations of the Lorentz group, Izv. Akad. Nauk SSSR, Ser. Mat. vol. 11, 1947, pp. 411–504.

---

* For a proof of this theorem, see [21], pp. 72–76. See also [20], pp. 251–254.

† See [5].

6. Paul R. Halmos, Finite-dimensional Vector Spaces, 2nd ed., Princeton, New Jersey, 1958.
7. Edwin Hewitt, Certain generalizations of the Weierstrass approximation theorem, Duke Math. J., vol. 14, 1947, pp. 419–427.
8. Edwin Hewitt, Integral representation of certain linear functionals, Ark. Mat., vol. 2, 1952, pp. 269–282.
9. Edwin Hewitt, Rings of real-valued continuous functions. I, Trans. Amer. Math. Soc., vol. 64, 1948, pp. 45–99.
10. John C. Holladay, A note on the Stone-Weierstrass theorem for quaternions, Proc. Amer. Math. Soc., vol. 8, 1957, pp. 656–657.
11. Shizuo Kakutani, Concrete representation of abstract (M)-spaces (a characterization of the space of continuous functions), Ann. of Math. (2), vol. 42, 1941, pp. 994–1024.
12. John L. Kelley, General Topology, New York, 1955.
13. Konrad Knopp, Theory and Application of Infinite Series (Translated from the German by Miss R. C. Young), London and Glasgow, 1928.
14. Casimir Kuratowski, Topologie I, 2nd ed., Monografie Matematyczne, vol. XX, Warszawa-Wrocław, 1948.
15. Henri Lebesgue, Sur l'approximation des fonctions, Bull. Sci. Math. (2), vol. 22, 1898, pp. 278–287.
16. Frédéric Riesz, Sur les opérations fonctionnelles linéaires, C. R. Acad. Sci. Paris, vol. 149, 1909, pp. 974–977.
17. Walter Rudin, Subalgebras of spaces of continuous functions, Proc. Amer. Math. Soc., vol. 7, 1956, pp. 825–830.
18. Stanislaw Saks, Theory of the Integral, 2nd ed., Monografie Matematyczne, vol. VII, Warszawa-Lwów, 1937.
19. Marshall H. Stone, Applications of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc., vol. 41, 1937, pp. 375–481.
20. Marshall H. Stone, The generalized Weierstrass approximation theorem, Math. Mag., vol. 21, 1948, pp. 167–183, 237–254.
21. André Weil, L'intégration dans les Groupes Topologiques et ses Applications, Act. Sci. et Ind. No. 869, Paris, 1940.
22. Kôsaku Yosida and Edwin Hewitt, Finitely additive measures, Trans. Amer. Math. Soc., vol. 72, 1952, pp. 46–66.

THE GENERALIZED COCONUT PROBLEM*

ROGER B. KIRCHNER, Harvard University

1. Introduction. The coconut problem has long interested students of mathematics. Accordingly, many methods have been applied to its solution, including congruences, properties of number bases, and elaborate sieves. In this paper, the calculus of finite differences is used to solve the following generalization.

Suppose that n sailors plan to divide a pile of coconuts in the morning. During the night, one of them decides to take his share. After discarding V_1 coconuts to make the division come out even, he takes one nth of the pile. The process is repeated by $m-2$ more sailors, the xth sailor discarding V_x coconuts.

* This work was done while holding a National Science Foundation Undergraduate Research Assistantship at Carleton College. The author acknowledges valuable assistance from Dr. K. O. May and Dr. J. M. Calloway in preparing this paper for publication.

In the morning, after discarding V_m coconuts, the remaining pile is divided equally. What is the minimum number of coconuts that could have been in the pile originally?

The traditional problem is the special case where $n=5$, $m=6$, and $V_i=1$, $i=1, 2, \cdots, 6$. A formula for the solution of the generalized problem will now be derived.

2. The difference equation and its solution. Abstracting from sailors and coconuts, we wish to find the smallest integer A such that for m nonnegative integers $V_1, \cdots, V_m$, there exist m positive integers $U_1, \cdots, U_m$, that satisfy

$$U_1 = \frac{A - V_1}{n}, \tag{1}$$

$$U_x = \frac{(n-1)U_{x-1} - V_x}{n}. \tag{2}$$

Since the V_i, $i>m$ do not affect the solution, we shall choose these such that $\Delta^k V_x = 0$, $k \geqq m$. This can always be done, and it will simplify the following equations.

The difference equation (2) with initial condition (1) is seen to be linear with constant coefficients, and defined for all positive x. Its solution is

$$U_x = \frac{(A - V_1 - nP_1)}{(n-1)}\left(\frac{n-1}{n}\right)^x + P_x, \tag{3}$$

where P_x is any particular solution to the nonhomogeneous equation. To determine P_x, we write the difference equation in the following equivalent form: $n\Delta U_x + U_x = -V_{x+1}$. A particular solution may be verified to be

$$P_x = -\sum_{k=0}^{m-1} (-n)^k \Delta^k V_{x+1}. \tag{4}$$

Using Newton's formula, V_{x+1} can be expressed in terms of the differences $\Delta^k V_1$, $k=0, 1, \cdots$, as

$$V_{x+1} = \sum_{j=0}^{m-1} \Delta^j V_1 C_j^x, \tag{5}$$

where

$$C_j^x = \begin{cases} \dfrac{x(x-1)\cdots(x-j+1)}{j!}, & j \geqq 0; \\ 0, & j < 0. \end{cases}$$

Noting that $\Delta^k C_j^x = C_{j-k}^x$, and that $C_j^x = 0$, $j<0$, we have upon substituting (5) into (4) and interchanging the order of summation:

$$P_x = -\sum_{j=0}^{m-1}\sum_{k=0}^{j}(-n)^k\Delta^j V_1 C_{j-k}^x. \tag{6}$$

We note that both V_x and P_x are integral when x is integral.

3. The determination of A. In order for A to be a solution, necessary conditions are that (i) A is integral, and (ii) U_k, $k=1, \cdots, m$, are positive integers. Therefore we must have

$$U_m = \frac{(A - V_1 - nP_1)}{(n-1)}\left(\frac{n-1}{n}\right)^m + P_m = r, \tag{7}$$

where r is an integer. Solving this equation for A in terms of r,

$$A = \frac{(r - P_m)n^m}{(n-1)^{m-1}} + V_1 + nP_1. \tag{8}$$

Since $n-1$ is relatively prime to n, and V_x and P_x are integers when x is an integer, a necessary and sufficient condition that A be integral is that

$$r - P_m = s(n-1)^{m-1} \tag{9}$$

for some integer s. From (2) we note that $U_i < U_{i-1}$ since $V_k \geqq 0$, $k=1, \cdots, m$. Thus if $U_m = r$ is positive then so will A and U_k, $k=1, \cdots, m$, be positive. To make A as small as possible, we choose the smallest integer s for which r is positive. This value of s is given by

$$s = \left[\frac{-P_m}{(n-1)^{m-1}}\right] + 1. \tag{10}$$

It is an easy matter to show that all the U_i are integral. From (7) and (9), we have $U_m = s(n-1)^{m-1} + P_m$. Using this and the original difference equation (2), solved for U_{x-1}, it follows by induction that

$$U_{m-k} = s(n-1)^{m-1-k}(n^k) + P_{m-k}, \qquad 0 \leqq k < m.$$

Therefore, for the choice of s in (10), conditions (i) and (ii) above hold. Furthermore, the A thus determined is the smallest compatible with the conditions on the U_i. Hence it is the solution. To obtain A explicitly, we substitute (9) and (10) into (8). Evaluating (6) at $x=1$ and $x=m$, and substituting these values into (8), we obtain

$$A = \left(\left[\frac{\sum_{j=0}^{m-1}\sum_{k=0}^{j}(-n)^k\Delta^j V_1 C_{j-k}^m}{(n-1)^{m-1}}\right] + 1\right)n^m + (1-n)\sum_{j=0}^{m-1}(-n)^j\Delta^j V_1. \tag{11}$$

If n and m are very large, the square bracket term will in general be difficult to calculate. However, this calculation may sometimes be avoided by noting that any two "solutions" will differ by a multiple of n^m. We can thus write

$$A = cn^m + (1 - n)\sum_{j=0}^{m-1} (-n)^j \Delta^j V_1, \tag{12}$$

where c is the smallest integer such that $A>0$, and $U_k>0$, $k=1, \cdots, m$. Fortunately, if n and m are larger than three or four, and the V_i are not too large, this value is usually the one between 0 and n^m. That is, A equals the right hand side of (12) modulo n^m.

It may be remarked that we have used the nonnegative character of the V_i only to establish that the U_i are positive. Therefore, with c defined as in (12), (12) is valid even if some of the V_i are negative.

To summarize, if a pile of coconuts is divided m times among n sailors in the manner described in the introduction, such that before the ith division, V_i coconuts are discarded, the minimum number of coconuts that could have been in the pile originally is given by formula (11) or (12).

4. Examples.

1. Consider the "generalization" where n sailors divide the pile m times and discard p coconuts before each division. Since $V_x \equiv p$, all the differences are zero. Formula (11) yields the solution

$$A = \left(\left[\frac{p}{(n-1)^{(m-1)}}\right] + 1\right) n^m + (1 - n)p.$$

If $0 \leqq p < (n-1)^{m-1}$, this simplifies to

$$A = n^m + (1 - n)p.$$

The classic coconut problem is the special case where $n=5$, $m=6$, and $p=1$. Substituting these values in, we find $A=15{,}621$.

2. This example illustrates the use of (12). Suppose $n=5$, $m=4$, $V_1=4$, $V_2=3$, $V_3=1$, $V_4=1$. The differences can be obtained from a difference table:

x	V_x	ΔV_x	$\Delta^2 V_x$	$\Delta^3 V_x$
1	4	−1	−1	3
2	3	−2	2	
3	1	0		
4	1			

We have $A=(1-n)\sum_{j=0}^{m-1}(-n)^j\Delta^j V_1 (\text{mod } n^m)=314$. This may be verified to be correct.

3. Finally, we use (12) in a case where the V_i are negative. This means that coconuts are being added to the pile (say by a monkey). Let $n=3$, $m=2$, $V_1=V_2=-10$. Then

$$A = 3^2c + (1 - 3)(-10) = 9c + 20, \quad U_1 = 3c + 10, \quad U_2 = 2c + 10.$$

Thus $c=-2$, and $A=2$.

ON THE BROCARD POINTS OF A TRIANGLE

PETER YFF, American University of Beirut

1. Preliminaries. Let (T) be any triangle $A_1A_2A_3$, with vertices numbered in counterclockwise order. The interior angle at A_i is denoted by α_i $(i=1, 2, 3)$, and the length of the opposite side is a_i.

The Brocard angle of (T) is called ω, for which one formula is $\cot\omega=\cot\alpha_1+\cot\alpha_2+\cot\alpha_3$. The inequality

$$\cot\omega \geqq \sqrt{3} \tag{1}$$

is well known, and the relationship

$$\cot\omega \leqq \frac{9}{8}\csc\alpha_1\csc\alpha_2\csc\alpha_3 \tag{2}$$

may be easily verified. In each case equality occurs only when (T) is equilateral.

Johnson [2] shows that, if ω is given, the shape of (T) is restricted by $\delta' \leqq \alpha_i \leqq \delta$ $(i=1, 2, 3)$, in which

$$\cot\delta = \tfrac{1}{2}(\cot\omega - 2\sqrt{\cot^2\omega - 3}), \tag{3}$$

$$\cot\delta' = \tfrac{1}{2}(\cot\omega + 2\sqrt{\cot^2\omega - 3}). \tag{4}$$

The value of δ or δ' is realized only by the vertex angle of an isosceles triangle.

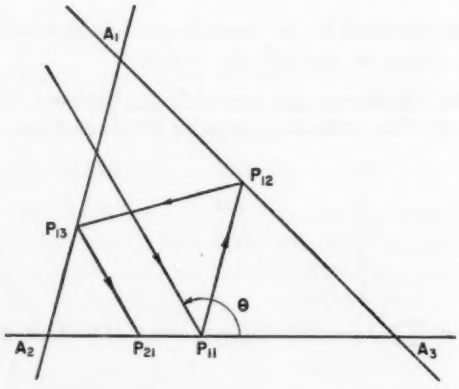


FIG. 1

2. A convergent path. From any point in the plane of (T) a line is drawn (Fig. 1) to meet A_2A_3 (extended if necessary) at a given angle $\theta(0<\theta<\pi)$, θ being measured counterclockwise from A_2A_3 to the line. From P_{11}, the point at which the line meets A_2A_3, a line is drawn to meet A_3A_1 at P_{12}, forming the angle θ in the

same sense as before. This path is continued indefinitely in the same manner, proceeding from one side of (T) to the next in counterclockwise order. It meets A_2A_3 for the nth time at P_{n1} $(n=1, 2, 3, \cdots)$, and the directed segment $P_{n1}A_3$ is denoted by u_n.

By trigonometry it is seen that $u_2=au_1+b$, in which

$$a = \frac{\sin(\alpha_1-\theta)\sin(\alpha_3-\theta)\sin(\alpha_3-\theta)}{\sin^3\theta} \quad \text{and} \quad b = \frac{a_1\sin^2\alpha_2}{\sin^2\theta}.$$

It follows that $u_n=a^{n-1}u_1+b(1+a+a^2+\cdots+a^{n-2})$. When $|a|<1$, $\lim_{n\to\infty} u_n$ exists and is equal to $b/(1-a)$. Clearly then the path converges to meet each side of (T) at a unique point.

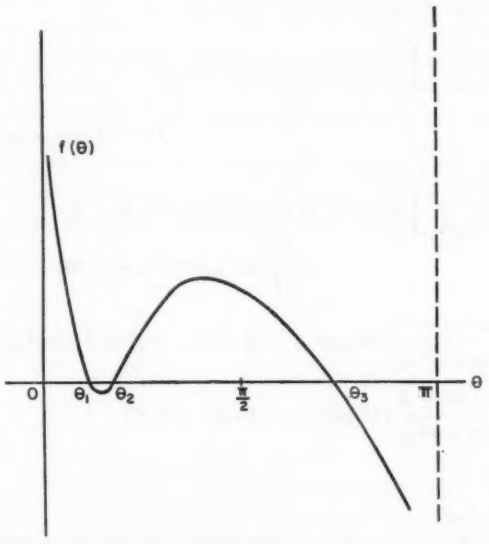


FIG. 2

The inequality $|a|<1$ may be reduced to $0<\csc^2\theta(\cot\omega-\cot\theta)<2\csc\alpha_1\csc\alpha_2\csc\alpha_3$, in which the left-hand inequality yields the result $\theta>\omega$. The other inequality may be written $f(\theta)>0$, if $f(\theta)=\cot^3\theta-\cot\omega\cot^2\theta+\cot\theta-\cot\omega+2\csc\alpha_1\csc\alpha_2\csc\alpha_3$. The essential properties of $f(\theta)$ are shown in Figure 2.

There are three values of θ for which $f(\theta)=0$. They are called θ_1, θ_2, θ_3 (in order of increasing magnitude when all are real). The root θ_3 is always real and lies within certain bounds which will now be determined.

When θ is near π, $f(\theta)<0$. But

$$f\left(\frac{2\pi}{3}\right) = -\frac{4}{3\sqrt{3}} - \frac{4}{3}\cot\omega + 2\csc\alpha_1\csc\alpha_2\csc\alpha_3$$

$$\geqq -\frac{4}{3\sqrt{3}} - \frac{4}{3}\cot\omega + \frac{16}{9}\cot\omega \quad \text{(from (2))}$$

$$= \frac{4}{9}(\cot\omega - \sqrt{3})$$

$$\geqq 0 \quad \text{(from (1))}.$$

Therefore $\theta_3 \geqq 2\pi/3$.

An upper bound may be obtained as follows. Given ω, $\csc\alpha_1\csc\alpha_2\csc\alpha_3$ becomes a function of one variable, and its extremes may be computed to be $(2/27)[\cot^3\omega + 9\cot\omega \pm (\cot^2\omega - 3)^{3/2}]$. Now

$$f(\pi - 2\omega) = -\frac{1}{8\cot^3\omega}\Big[3\cot^6\omega + 5\cot^4\omega + \cot^2\omega - 1 - 16\cot^3\omega\csc\alpha_1\csc\alpha_2\csc\alpha_3\Big]$$

$$\leqq -\frac{1}{8\cot^3\omega}\Big[3\cot^6\omega + 5\cot^4\omega + \cot^2\omega - 1 - \frac{32}{27}\cot^3\omega\{\cot^3\omega + 9\cot\omega + (\cot^2\omega - 3)^{3/2}\}\Big]$$

$$= -\frac{\cot^2\omega - 3}{216\cot^3\omega}[49\cot^4\omega - 6\cot^2\omega + 9 - 32\cot^3\omega(\cot^2\omega - 3)^{1/2}]$$

$$\leqq -\frac{\cot^2\omega - 3}{216\cot^3\omega}(17\cot^4\omega - 6\cot^2\omega + 9)$$

$$\leqq 0.$$

Therefore $\theta_3 \leqq \pi - 2\omega$.

In case θ_1 and θ_2 are imaginary, the condition for convergence of the path is stated as $\omega < \theta < \theta_3$. However, they may be real. Several facts which may be derived will now be stated without proof.

The relative minimum and maximum of $f(\theta)$ are at $\theta = \frac{1}{2}(\pi - \delta)$ and $\theta = \frac{1}{2}(\pi - \delta')$ respectively. The relative minimum has the same sign as $\theta_3 - \delta$; hence θ_1 and θ_2 are real when $\theta_3 \leqq \delta$. If θ_1 and θ_2 are real, then $\theta_1 + \theta_2 + \theta_3 = \pi$ and $\cot\theta_1 + \cot\theta_2 + \cot\theta_3 = \cot\omega$; this means that $\theta_1, \theta_2, \theta_3$ are the angles of a triangle with the same Brocard angle as that of (T).

Also it is easily shown that $\theta_1 > \omega$. Therefore, when θ_1 and θ_2 are real, the domain of convergence of the path is separated into two disjoint intervals: $\omega < \theta < \theta_1$ and $\theta_2 < \theta < \theta_3$.

In particular, the path converges when $\theta = \alpha_i$ $(i = 1, 2, 3)$. If we assume

$\alpha_1 \leqq \alpha_2 \leqq \alpha_3$, then $\alpha_1 \leqq \frac{1}{2}(\pi - \delta) \leqq \alpha_2$. Hence, in the case for which θ_1 and θ_2 are real, α_1 lies in the first subset of the domain of convergence, while α_2 and α_3 lie in the second subset.

3. A convergent sequence of triangles. Whether or not the path converges, there always exists, with one exception, a triangle inscribed in (T) each of whose sides makes the angle θ with the corresponding side of (T). This triangle is also similar to (T), the ratio of similitude being $\sin \omega / \sin (\theta - \omega)$. Thus the exceptional case is that for which $\theta = \omega$.

These inscribed triangles are known to be the Miquel triangles of Ω_2, the second Brocard point of (T), which is also the center of similitude of (T) and each inscribed triangle [2, p. 269].

Let (T_1) be the inscribed triangle for a given value of θ. Then inscribe (T_2) in (T_1) in the same way that (T_1) was inscribed in (T). Continue this process to obtain an infinite sequence of triangles. This sequence converges to a point if $|\sin \omega / \sin (\theta - \omega)| < 1$, from which it follows that $2\omega < \theta < \pi$. The limit point is Ω_2, since it is the center of similitude of every pair of successive triangles and is always an interior point.

If, from the beginning, θ had been measured clockwise and the path had been defined as going in clockwise order from each side of (T) to the next, the limit point of the resulting sequence of triangles would have been Ω_1, the first Brocard point of (T).

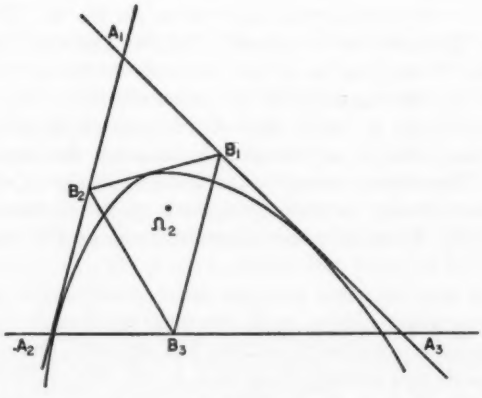


FIG. 3

4. Three confocal parabolas. Now let (T) be the triangle of reference in a trilinear homogeneous coordinate system; and let triangle $B_1B_2B_3$ be the inscribed triangle corresponding to θ, as described above (Fig. 3). Then the coordinates of its vertices are as follows:

$$B_1(a_2 \sin(\theta - \alpha_3), 0, a_3 \sin\theta),$$
$$B_2(a_1 \sin\theta, a_3 \sin(\theta - \alpha_1), 0),$$
$$B_3(0, a_2 \sin\theta, a_1 \sin(\theta - \alpha_2)).$$

The equation of the line B_1B_2 may now be derived. When it is differentiated with respect to θ, and θ is eliminated, the equation of the envelope is obtained:

$$\left(x_1 - \frac{a_2 \sin(\alpha_1 - \alpha_3)}{a_3 \sin\alpha_1} x_3\right)^2 - \frac{4a_2}{a_3} x_2x_3 = 0. \tag{5}$$

Thus the envelope of B_1B_2, as θ varies, is a conic. Solving its equation simultaneously with that of the line at infinity, $a_1x_1+a_2x_2+a_3x_3=0$, we learn that it is tangent to this line and is therefore a parabola. This parabola is tangent to A_1A_2 at A_2 and is also tangent to A_1A_3.

The parabolic envelopes of B_1B_3 and B_2B_3 are obtained by cyclic permutation of the subscripts in (5). It will now be shown that the three parabolas have a common focus.

The envelope of B_1B_2 is tangent to each side of triangle $A_1B_2B_1$. Therefore its focus is known to be on the circumcircle of this triangle for every value of θ. The trilinear coordinates of Ω_2 with respect to triangles $A_1A_2A_3$ and $B_1B_2B_3$ are $(a_2/a_3, a_3/a_1, a_1/a_2)$. With respect to triangle $A_1B_2B_1$, the second and third coordinates are still in the same ratio, since this triangle has two sides in common with triangles $A_1A_2A_3$. Now the distance of Ω_2 from B_1B_2 equals its distance from A_1A_2 multiplied by the ratio of similitude $\sin\omega/\sin(\theta-\omega)$. This distance is also negative because Ω_2 is exterior to triangle $A_1B_2B_1$. Therefore its coordinates are $(-a_1 \sin\omega/[a_2 \sin(\theta-\omega)], a_3/a_1, a_1/a_2)$. Since the angles of triangle $A_1B_2B_1$ are α_1, $\pi-\theta$, and $\theta-\alpha_1$, the equation of its circumcircle is $y_2y_3 \sin\alpha_1+y_3y_1 \sin\theta+y_1y_2 \sin(\theta-\alpha_1)=0$ [3, p. 357]. The coordinates of Ω_2 satisfy this equation identically in θ, and thus it is proved that Ω_2 is on the circumcircle of every triangle $A_1B_2B_1$. Therefore it must be the focus of the parabola enveloping the lines B_1B_2. Similarly it may be shown that Ω_2 is on the circumcircles of triangles $A_2B_3B_2$ and $A_3B_1B_3$. Thus it is the common focus of the three parabolic envelopes.

At this time it may be noted that the usual construction of a Brocard point involves three concurrent circles, each one passing through two vertices of the triangle and tangent to one side [1, p. 274]. For example, one such circle through Ω_2 passes through A_2 and is tangent to A_1A_3 at A_1. But the foregoing discussion has shown that this circle is the limiting case, as θ approaches zero, of the coaxal system through A_1 and Ω_2.

Finally, the relative sizes of the three parabolas may be determined. Triangle $B_1B_2B_3$ is smallest when $|\sin\omega/\sin(\theta-\omega)|$ is minimum, that is, when $\theta=\omega+\frac{1}{2}\pi$. This is the pedal triangle of Ω_2. For this value of θ each side of $B_1B_2B_3$ is nearest the focus of its parabola; in other words, each side is tangent at the vertex. It follows that the distance from focus to vertex in each parabola is

proportional to the corresponding trilinear coordinate of Ω_2.

Similar results may be obtained for Ω_1, the first Brocard point.

References

1. N. Altshiller-Court, College Geometry, 2nd ed., 1952, pp. 274–284.
2. R. A. Johnson, Modern Geometry, Boston, 1929, pp. 263–289.
3. C. Smith, Conic Sections, London, 1924, pp. 341–365.

FIBONACCI SERIES MODULO m

D. D. WALL, IBM Corporation

This inquiry is concerned with determining the length of the period of the recurring series obtained by reducing a Fibonacci series by a modulus m. The problem arose in connection with a method for generating random numbers, but it turned out to be unexpectedly intricate, and so quickly became of interest in its own right. The function studied (length of period as a function of starting values and modulus) exhibits quite a few apparent properties which are established here. At least two questions remain unanswered: see remarks after Theorems 5 and 7.

Let f_n denote the nth member of the Fibonacci series $f_0=a$, $f_1=b$, $f_{n+1}=f_n+f_{n-1}$. We reduce f_n modulo m, taking least nonnegative residues, and let h denote the length of the period of the repeating series that results. The letter p is reserved to designate a prime, but a, b, and m may be arbitrary integers, except that we assume, without loss of generality, that a, b, and m are relatively prime: $(a, b, m)=1$. We will also refer to the two special Fibonacci series u_n and v_n defined by $u_0=0$, $u_1=1$, and $v_0=2$, $v_1=1$, and will make use of many of the known properties of these series. It will be convenient to let $k=k(m)$ denote the length of the period of $u_n \pmod m$, in distinction from h which depends on a and b as well as m.

Example: The values of $u_n \pmod 7$ are 0, 1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1, and then repeat; so $k(7)=16$. We are curious as to the relation between the number 16 and the number 7. Note that $u_8 \equiv 0 \pmod 7$ so the 16 terms in the period form two sets of 8 terms each, the terms of the second half being 6, or -1, times the corresponding terms of the first half. Theorem 7 generalizes this property and explains the relation $k(7)=16$.

THEOREM 1. *$f_n \pmod m$ forms a simply periodic series. That is, the series is periodic and repeats by returning to its starting values.*

Proof. The series repeats because there are only a finite number m^2 of pairs of terms possible, and the recurrence of a pair results in recurrence of all following terms. From the defining relation we have $f_{n-1}=f_{n+1}-f_n$, so if $f_{t+1}\equiv f_{s+1}$ and $f_t\equiv f_s$, (mod m), then $f_{t-1}\equiv f_{s-1}$, $\cdots$, $f_{t-s+1}\equiv f_1$, and $f_{t-s}\equiv f_0$, so that the series is simply periodic.

COROLLARY. $u_k\equiv 0 \pmod m$. (A direct consequence of Theorem 1.)

THEOREM 2. *If m has the prime factorization $m=\prod p_i^{e_i}$ and if h_i denotes the length of the period of $f_n \pmod{p_i^{e_i}}$, then $h=\operatorname{lcm}[h_i]$, the least common multiple of the h_i.*

Proof. The statement, "h_i is the length of the period of $f_n \pmod{p_i^{e_i}}$," implies that the series $f_n \pmod{p_i^{e_i}}$ repeats only after blocks of length ch_i; and the statement, "h is the length of the period of $f_n \pmod m$," implies that $f_n \pmod{p_i^{e_i}}$ repeats after h terms for all values of i. Therefore h is of the form ch_i for all values of i, and since any such number gives a period of $f_n \pmod m$, we conclude that $h=\operatorname{lcm}[h_i]$.

In view of Theorem 2, we may henceforth assume that m is of the form $m=p^e$.

The next five theorems establish properties of the special series u_n with $u_0=0$, $u_1=1$; k is the length of the period of this series, mod m. A close relationship between the special series u_n and the more general series f_n is contained in the formula $f_n=bu_n+au_{n-1}$, which shows that f_n repeats after k terms, so that h is a divisor of k: $h \mid k$.

THEOREM 3. *The terms for which $u_n\equiv 0 \pmod m$ have subscripts that form a simple arithmetic progression. That is, $n=xd$ for $x=0, 1, 2, \cdots$, and some positive integer $d=d(m)$, gives all n with $u_n\equiv 0 \pmod m$.*

Proof. From the known relations $(u_n, u_{n+1})=1$ and $u_{n+t}=u_{n+1}u_t+u_nu_{t-1}$, we see that $u_i\equiv 0 \pmod m$ and $u_j\equiv 0 \pmod m$ imply $u_{i+j}\equiv 0 \pmod m$ and (with $i\geqq j$) $u_{i-j}\equiv 0 \pmod m$. The first follows by setting $n=i$, $t=j$; and setting $n+t=i$, $n=j$ gives $u_{n+1}u_t\equiv 0 \pmod m$, which with $(u_n, u_{n+1})=1$ along with $u_n\equiv 0 \pmod m$ gives the second congruence $u_t=u_{i-j}\equiv 0 \pmod m$. Therefore, the subscripts n that we are concerned with comprise the nonnegative terms of a module, and so are of the form $n=xd$. The corollary to Theorem 1 shows that u_0 is not the only $u_n\equiv 0 \pmod m$, so $d>0$, and this completes the proof of the theorem.

Remark. We note that $d \mid k$. Empirically we find many m with $d=k$, but also many with $d<k$.

THEOREM 4. *If $m>2$ then k is an even number.*

Proof. Suppose that k is odd: $k=2x+1$. Then by working both ends to the middle with the defining relation, all congruences being mod m, we find:

$$-u_k \equiv 0 = u_0, \quad u_{k-1} \equiv 1 = u_1, \quad -u_{k-2} = -u_k + u_{k-1} \equiv u_0 + u_1 \equiv u_2,$$

. . . . . .

$$(-1)^{t-1}u_{k-t} = (-1)^{t-1}u_{k-t+2} + (-1)^t u_{k-t+1} \equiv u_{t-2} + u_{t-1} = u_t,$$

. . . . . .

$$(-1)^{x-2}u_{x+2} \equiv u_{x-1}, \qquad (-1)^{x-1}u_{x+1} \equiv u_x.$$

From this last congruence and the defining relation, it follows that $u_{x-1} \equiv 0 \pmod m$ if x is odd, and $u_{x+2} \equiv 0 \pmod m$ if x is even; but then the congruence just prior to the last one in the sequence shows that again $u_{x-1} \equiv 0 \pmod m$.

Now from Theorem 3, $d \mid (x-1)$ so $d \mid (2x-2)$, also $d \mid k$ so $d \mid (2x+1)$; therefore, $d \mid (2x+1-2x+2)$ so $d=3$. Finally $u_d = u_3 = 2 \equiv 0 \pmod m$ shows $m=2$ is implied by the hypothesis that k is odd. Therefore for $m>2$, k must be even.

THEOREM 5. *If $k(p^2) \neq k(p)$, then $k(p^e) = p^{e-1}k(p)$. Also, if t is the largest integer with $k(p^t) = k(p)$, then $k(p^e) = p^{e-t}k(p)$ for $e>t$.*

Proof. By solving the standard formulas $u_n = (r^n - s^n)/\sqrt{5}$ and $v_n = r^n + s^n$ for r^n and s^n in terms of u_n and v_n (where r and s satisfy $x^2 = x+1$), expanding $(r^n)^a$ and $(s^n)^a$ by the binomial theorem, and recombining, we obtain the relations

$$u_{an} = (r^{an} - s^{an})/\sqrt{5} = [2^{-a}(\sqrt{5}u_n + v_n)^a - 2^{-a}(-\sqrt{5}u_n + v_n)^a]/\sqrt{5}$$

$$= 2^{1-a} \sum_{j \text{ odd}} \binom{a}{j} 5^{(j-1)/2} u_n^j v_n^{a-j} = 2^{1-a} u_n (K u_n^2 + a v_n^{a-1}),$$

where K is an integer, and similarly

$$u_{an+1} = 5^{-1/2}2^{-a} \sum_0^a \binom{a}{j} 5^{j/2} u_n^j v_n^{a-j} \left[\frac{1+\sqrt{5}}{2} - (-1)^j \frac{1-\sqrt{5}}{2}\right]$$

$$= 2^{-a}(K u_n^2 + a u_n v_n^{a-1} + v_n^a),$$

where K is an integer. The theorem follows from these relations by induction on e, except for the case $p=2$. We will outline the induction step, observing that the theorem is trivially true for $e=1$ ($e=t$ in the second statement), which enables the induction to begin.

On noting that $v_n = u_{n-1} + u_{n+1}$ has a g.c.d. with u_n of either 1 or 2, and applying Theorem 3, the first formula above gives the "law of repetition of primes": *if u_n is the first term $\equiv 0 \pmod{p^e}$ but $\not\equiv 0 \pmod{p^{e+1}}$, then u_{pn} is the first term $\equiv 0 \pmod{p^{e+1}}$, also $u_{pn} \not\equiv 0 \pmod{p^{e+2}}$.* For this value of n, the terms u_{nx} for $x = 0, 1, 2, \cdots$ are the terms $\equiv 0 \pmod{p^e}$, and u_{pnx} gives all terms $\equiv 0 \pmod{p^{e+1}}$. On setting $u_{pnx+1} \equiv 1 \pmod{p^{e+1}}$, to obtain $k(p^{e+1}) = pnx$, the second formula above gives

$$(v_{nx}/2)^p \equiv 1 \pmod{p^{e+1}},$$

$$v_{nx} \equiv 2 \pmod{p^e}, \qquad u_{nx+1} \equiv 1 \pmod{p^e},$$

so $k(p^e)=nx$ and $k(p^{e+1})=pk(p^e)$. For the exceptional case $p=2$ we use the formulas $u_{2n}=u_n(u_{n-1}+u_{n+1})$ and $u_{2n+1}=u_{n+1}^2+u_n^2$ to establish by induction that $k(2^e)=2^{e-1}k(2)$; the details are omitted here.

Remark. The most perplexing problem we have met in this study concerns the hypothesis $k(p^2)\neq k(p)$. We have run a test on a digital computer which shows that $k(p^2)\neq k(p)$ for all p up to 10,000; however, we cannot yet prove that $k(p^2)=k(p)$ is impossible. The question is closely related to another one, "can a number x have the same order mod p and mod p^2?", for which rare cases give an affirmative answer (*e.g.*, $x=3$, $p=11$; $x=2$, $p=1093$); hence, one might conjecture that equality may hold for some exceptional p.

THEOREM 6. *If* $m=p=10x\pm1$, *then* $k(p)\,|\,(p-1)$.

LEMMA. *The congruence* $x^2\equiv x+1 \pmod p$ *has a double root only for* $p=5$.

Proof of Lemma. $x^2-x-1\equiv(x-r)^2$ implies $2r\equiv1$ and $r^2\equiv-1$; these give $4r^2\equiv1$ and $4r^2\equiv-4$, so $5\equiv0$ which implies a modulus of 5.

Proof of Theorem 6. The number 5 is a quadratic residue for primes of the form $p=10x\pm1$, so the congruence $x^2\equiv x+1 \pmod p$, which is equivalent to $(2x-1)^2\equiv5 \pmod p$, has distinct roots r and s; therefore $u_n\equiv(r^n-s^n)/(r-s) \pmod p$.

Let g denote the lcm of the order of $r \pmod p$ and the order of $s \pmod p$. Since $rs\equiv-1 \pmod p$, we see that g is equal to the order of r if this number is even, and otherwise is equal to twice the order of r. In either case, $u_n \pmod p$ repeats after g terms, so $k(p)\,|\,g$, and Fermat's theorem shows $g\,|\,(p-1)$, so the theorem is proved.

THEOREM 7. *If* $m=p=10x\pm3$, *then* $k(p)\,|\,(2p+2)$.

Proof. The number 5 is a quadratic nonresidue for primes of this form, so $5^{(p-1)/2}\equiv-1 \pmod p$. We let $n=p$ and then let $n=p+1$ in the known formula

$$u_n = 2^{1-n}\left[\binom{n}{1}+5\binom{n}{3}+5^2\binom{n}{5}+\cdots\right].$$

The first substitution gives

$$u_p \equiv 5^{(p-1)/2}\binom{p}{p} \equiv -1 \pmod p,$$

and the second gives

$$u_{p+1} \equiv 2^{-1}\left[\binom{p+1}{1}-\binom{p+1}{p}\right] \equiv 0 \pmod p.$$

These congruences and the defining relation show that $u_{p+2}\equiv-u_1$, $u_{p+3}\equiv-u_2$, $\cdots$, $u_{2p+1}\equiv-u_p\equiv1$, $u_{2p+2}\equiv-u_{p+1}\equiv0 \pmod p$. Therefore, $u_n \pmod p$ repeats beginning with $u_{2p+2}\equiv u_0$, so that $k(p)\,|\,(2p+2)$.

COROLLARY. *If* $p=10x\pm3$ *then* $k(p)\equiv0 \pmod 4$.

Proof. Otherwise $k(p)$ would divide $(p+1)$ which would imply $u_p\equiv+1$.

Remarks. Theorems 6 and 7 furnish upper bounds for the function $k(p)$, and we easily find cases where $k(p)$ has these maximum values. On the other hand, we cannot find a nontrivial lower bound for $k(p)$, and a table of values of $k(p)$ shows many entries with $k(p)$ smaller than the upper limits of Theorems 6 and 7. Examples: $k(491)=490$, $k(521)=26$, $k(4993)=9988$, $k(9349)=38$.

We now return to the more general series f_n. One of the most interesting properties of h, the length of the period of $f_n \pmod m$, is that it is often independent of the starting values a and b. Theorems 8–12 describe this property. In such cases we will write $h=h(m)$, and relate h to $k(m)$, the length of the period of $u_n \pmod m$.

THEOREM 8. *If* $p=10x\pm3$, *then* $h(p^e)=k(p^e)$.

Proof. The congruences which indicate that $f_n \pmod m$ repeats with period h may be written in the following form:

$$f_h - a = bu_h + a(u_{h-1} - 1) \equiv 0 \pmod m,$$
$$f_{h+1} - b = (b + a)u_h + b(u_{h-1} - 1) \equiv 0 \pmod m.$$

Considering a and b as given coefficients, the determinant of this system is $D=b^2-ab-a^2$. With $m=p^e$, if $D\equiv0 \pmod p$ then $4a^2+4ab+b^2=(2a+b)^2 \equiv5b^2 \pmod p$; and $b\not\equiv0 \pmod p$ simultaneously with $D\equiv0 \pmod p$ since this would imply $a\equiv0 \pmod p$, contradicting $(a, b, m)=1$. Therefore $D\equiv0 \pmod p$ implies that 5 is a quadratic residue of p. But 5 is not a quadratic residue of primes $p=10x\pm3$, so for these p and $m=p^e$ we have $(D, m)=1$ and the only solution to the system of congruences is $u_h\equiv0$, $u_{h-1}\equiv1 \pmod{p^e}$, which shows that $k \mid h$. Since also $h \mid k$, we therefore have $h=k$, and the theorem is proved.

COROLLARY. *Whenever* $D=b^2-ab-a^2$ *satisfies* $(D, m)=1$, *then* $h=k$. *In particular,* v_n *has* $D=-5$ *so if* $(5, m)=1$, *then the length of the period of* $v_n \pmod m$ *is* $k(m)$.

THEOREM 9. *If* $m=2^e$, *then* $h=k$, *and if* $m=5^e$, *then either* $h=k$ *or else* $h=(1/5)k$, *according as* $D=b^2-ab-a^2$ *is not or is divisible by* 5.

Proof. As in Theorem 8, we examine the determinant D. By observing the three cases with $(a, b, 2)=1$, we see that $D\not\equiv0 \pmod 2$, so $(D, 2^e)=1$ and $h=k$ for $m=2^e$. If a and b give $D\not\equiv0 \pmod 5$, then $h=k$ for $m=5^e$. Although it is possible to have $D\equiv0 \pmod 5$, the congruence $(2a+b)^2\equiv5b^2 \pmod m$, which is equivalent to $D\equiv0 \pmod m$, shows that $D\not\equiv0 \pmod{5^2}$. In this case the congruences $f_h-a\equiv0 \pmod{5^e}$ and $f_{h+1}-b\equiv0 \pmod{5^e}$, from Theorem 8, give $u_h\equiv0 \pmod{5^{e-1}}$ and $u_{h+1}\equiv1 \pmod{5^{e-1}}$, so $k(5^{e-1}) \mid h(a, b, 5^e)$ and $h(a, b, 5^e)$ $=$ either $k(5^e)$ or $(1/5)k(5^e)$. But the second value always holds, for $D\equiv0 \pmod 5$ implies $b=-2a+5t$, and the formula

$$u_n = 2^{1-n}\left[\binom{n}{1} + 5\binom{n}{3} + 5^2\binom{n}{5} + \cdots\right]$$

shows that $u_{(1/5)k(5^e)} = u_{4\cdot 5^{e-1}} \equiv 3\cdot 5^{e-1} \pmod{5^e}$ and $u_{4\cdot 5^{e-1}+1} \equiv 1+4\cdot 5^{e-1} \pmod{5^e}$, since the terms $5\binom{n}{3}$, $5^2\binom{n}{5}$, $\cdots$, are all divisible by 5^e. Therefore $f_{(1/5)k(5^e)} \equiv (-2a+5t)\cdot 3\cdot 5^{e-1} + a(1+5^{e-1}) \equiv a \pmod{5^e}$, and $f_{(1/5)k(5^e)+1} \equiv (-2a+5t)\cdot(1+4\cdot 5^{e-1}) + a(3\cdot 5^{e-1}) \equiv b \pmod{5^e}$. These formulas require $e>1$, but for, $e=1$, an enumeration of cases shows $k=20$ and $h=20$ for $D \not\equiv 0 \pmod 5$ and $h=4$ when $D\equiv 0 \pmod 5$, so the theorem is proved.

COROLLARY. *If $m=p^e$ and h is odd, then $p=10x\pm 1$ or $m=2$.*

(A direct consequence of Theorems 4, 8, and 9.)

THEOREM 10. *If $m=p^e$, $p>2$, and if (a, b) give $h=2t+1$, then $k=4t+2$.*

Proof. The congruences which indicate that $f_n \pmod m$ repeats with period h may also be written as follows:

$$bu_h + a(u_{h-1} - 1) \equiv 0 \pmod m,$$
$$b(u_{h+1} - 1) + au_h \equiv 0 \pmod m.$$

The condition $(a, b, m)=1$ implies that $u_h^2-(u_{h+1}-1)(u_{h-1}-1)\equiv 0 \pmod m$, and the known formula $u_h^2-u_{h+1}u_{h-1}=(-1)^{h-1}$ permits this congruence to be simplified to $u_{h+1}+u_{h-1}=v_h=u_{2h}/u_h\equiv 1+(-1)^h \pmod m$. Therefore, h odd implies $u_{2h}\equiv 0 \pmod m$. Since $f_n \pmod m$ also repeats with period $2h$, the analogous congruences stating this fact lead to the analogous condition $u_{2h+1}+u_{2h-1} \equiv 1+(-1)^{2h}=2 \pmod m$, so $u_{2h}=0 \pmod m$ further implies $2u_{2h+1}\equiv 2$, $u_{2h+1} \equiv 1 \pmod m$, so $k \mid 2h$. Finally, k is even since $m>2$, so h odd requires $k=2h$, and the theorem is proved.

Examples:

$m = 11$, $a = 1$, $b = 4$ gives $h = 5$, while $k(11) = 10$;

$m = 29$, $a = 1$, $b = 24$ gives $h = 7$, while $k(29) = 14$;

$m = 121$, $a = 1$, $b = 37$ gives $h = 55$, while $k(121) = 110$.

THEOREM 11. (Converse to Theorem 10.) *If $m=p^e$, $p>2$, and if $k=4t+2$, then $h=2t+1$ for some (a, b).*

LEMMA. *If $k(p^e)=4t+2$, then $u_{2t+2}\equiv -u_{2t} \pmod{p^e}$.*

Proof of Lemma. This condition follows directly from the chain of congruences developed in Theorem 4, k now being even.

Proof of Theorem 11. Let

$$a = f_0 \equiv -u_{2t+1} - u_0 \pmod{p^e}, \qquad b = f_1 \equiv u_{2t} - u_1.$$

Then $f_n \equiv (-1)^{n-1}u_{2t+1-n} - u_n$, so $f_{2t+1} \equiv f_0$ and $f_{2t+2} \equiv -u_{-1} - u_{2t+2} \equiv -u_1 + u_{2t} = f_1$. Therefore $h \mid (2t+1)$; but then Theorem 10 shows $k = 2h$. This completes the proof except it must be verified that $(a, b, p^e) = 1$, which follows from Theorem 4 since $a \equiv -u_{2t+1} - u_0 \equiv b \equiv u_{2t} - u_1 \equiv 0 \pmod{p}$ would give $k(p) = 2t+1$, which is impossible. Incidentally, the case $p^e = 4$ is actually an exceptional case, since $k(4) = 6 \equiv 2 \pmod 4$, but no series $f_n \pmod 4$ has $h = 3$.

THEOREM 12. *If $m = p^e$, $p > 2$, $p \neq 5$, and h is even, then $h = k$.*

Proof. We use the condition $v_h \equiv 1 + (-1)^h \pmod m$, from Theorem 10, and the relation $v_n = r^n + s^n$ where r and s are the real roots of the equation $x^2 = x + 1$. Then, since h is even, and since $rs = -1$,

$$r^h + s^h - 2 \equiv 0 \pmod m,$$

$$r^{2h} + s^{2h} + 4 + 2(rs)^h - 4r^h - 4s^h \equiv 0 \pmod{m^2},$$

$$r^{2h} - 2 + s^{2h} \equiv (r^h - s^h)^2 \equiv 0 \pmod{m^2}.$$

Now $r^h - s^h$ is not an integer, but is of the form $x\sqrt{5}$; and since $p \neq 5$ assures $5 \not\equiv 0 \pmod m$, we may divide by $5 = (r-s)^2$ to obtain

$$\left[(r^h - s^h)/(r - s)\right]^2 = u_h^2 \equiv 0 \pmod{m^2}, \qquad u_h \equiv 0 \pmod m.$$

Finally $u_h \equiv 0$ and $v_h \equiv 2$ imply $u_{h-1} \equiv u_{h+1} \equiv 1$, which in turn implies $h = k$.

COROLLARY 1. *If $p \neq 5$ and $k(p) \equiv 0 \pmod 4$, then $h(p^e) = k(p^e)$.* (A direct consequence of Theorems 10 and 12, except for $p = 2$ which is covered by Theorem 9.)

COROLLARY 2. *If $h(a, b, p) = k(p)$ and $k(p^2) \neq k(p)$, then $h(a, b, p^e) = k(p^e)$.*

Proof. $h(a, b, p^e)$ must be a multiple of $h(a, b, p) = k(p)$, and must be a divisor of $k(p^e) = p^{e-1}k(p)$, hence must be of the form $p^{-c}k(p^e)$ for $c \geqq 0$. But Theorems 10 and 12 show that $h = k$ or $h = k/2$, so $h(a, b, p^e) = k(p^e)$. The cases $p = 2$ and $p = 5$ are covered by Theorem 9; for $p = 5$ note that $h = k$ or $h = (1/5)k$ independently of e.

Remark. The converse of Corollary 2 is false. For example, $h(3, 1, 11) < k(11)$, but $h(3, 1, 11^2) = k(11^2)$.

Example. Find h and k for $m = 10^{10}$.

As per Theorem 2, we consider $m_1 = 2^{10}$ and $m_2 = 5^{10}$. To apply Theorem 5, we check that $k(2) = 3 \neq k(2^2)$ and $k(5) = 20 \neq k(5^2)$. Therefore $k_1 = 3.2^9$, $k_2 = 20.5^9$, $k = \text{lcm}\,[k_1, k_2] = 15.10^9$. Finally, Theorem 9 shows that $h = 15.10^9$ or $h = 3 \cdot 10^9$ according as $D = b^2 - ab - a^2$ is not or is divisible by 5.

Remark. We note in passing that the number of ordered pairs $(a, b) \pmod m$ with $(a, b, m) = 1$ is given by the formula $m^2 \prod_{p \mid m} (1 - 1/p^2)$.

Since the results of this study direct interest to the function $k(p)$, we append a table of $k(p)$, listing all cases with $5<p<2000$ for which $k(p)$ is smaller than the maximum value permitted by Theorems 6 and 7.

p	$k(p)$	p	$k(p)$	p	$k(p)$
29	14	743	496	1279	426
47	32	761	380	1289	322
89	44	769	192	1291	430
101	50	797	228	1307	872
107	72	809	202	1361	680
113	76	811	270	1381	460
139	46	829	276	1409	704
151	50	859	78	1427	168
181	90	881	176	1471	490
199	22	911	70	1483	424
211	42	919	102	1511	302
229	114	941	470	1523	1016
233	52	953	212	1549	774
263	176	967	176	1553	1036
281	56	977	652	1579	526
307	88	991	198	1597	68
331	110	1009	126	1601	160
347	232	1021	510	1621	810
349	174	1031	206	1669	834
353	236	1049	262	1699	566
401	200	1061	530	1709	854
421	84	1069	356	1721	430
461	46	1087	128	1733	1156
509	254	1097	732	1741	870
521	26	1103	96	1789	894
541	90	1109	554	1823	1216
557	124	1151	230	1861	930
563	376	1217	812	1871	374
619	206	1223	816	1877	1252
661	220	1229	614	1913	1276
677	452	1231	410	1951	390
691	138	1249	624	1973	1316
709	118	1277	852	1999	666

The author wishes to acknowledge the assistance of George E. Collins, who carefully reviewed the paper and made many helpful suggestions.

Reference

L. E. Dickson, History of the Theory of Numbers, vol. 1. Washington, D. C., 1920.

MATHEMATICAL INDUCTION IN COMMUTATIVE SEMIGROUPS†

SETH WARNER, Duke University

1. The principle of mathematical induction in commutative semigroups. One formulation of the First Principle of Mathematical Induction is that if a subset S of the positive integers N^* contains 1 and also contains $n+1$ whenever it contains n, then $S=N^*$. The set N^*, with addition, is an example of a semigroup, that is, a set furnished with a composition $+$ satisfying $(x+y)+z=x+(y+z)$ for all members x, y, z of the set. It is further a commutative semigroup, that is, one satisfying $x+y=y+x$ for all elements x and y. We shall say that a semigroup D with composition $+$ satisfies (PMI) if there exists $\beta\in D$ such that for any subset S of D, if $\beta\in S$ and if $x+\beta\in S$ whenever $x\in S$, then $S=D$. (We shall see that any such semigroup is necessarily commutative.) The natural numbers (nonnegative integers) N form a commutative semigroup which does not satisfy (PMI), the proper inductive statement for N being, of course, that if subset S of N contains 0 and contains $n+1$ whenever it contains n, then $S=N$. Analogously, we shall say that a commutative semigroup D with composition $+$ satisfies (PMI)′ if there exist members α and β of D such that for any subset S of D, if $\alpha\in S$ and if $x+\beta\in S$ whenever $x\in S$, then $S=D$. It is natural to ask which commutative semigroups, in addition to N^*, satisfy (PMI), and which ones, in addition to N^* and N, satisfy (PMI)′. We shall see that it is a simple matter to give a complete description of all such semigroups.

2. Dipper semigroups. For any natural number n and any positive integer m let H_n be the set of integers k satisfying $0\leqq k\leqq n$, H_n^* the set of integers k satisfying $1\leqq k\leqq n$ (H_0^* is thus the empty set), and let C_m be the cyclic group of the integers modulo m under addition. For any integer k, we let k_m be the integer modulo m determined by k. Let $D(n, m)$ be the set $H_n\cup C_m$. We define a composition $+_{n,m}$ on $D(n, m)$ as follows: If x, $y\in H_n$ and $x+y\leqq n$, then $x+_{n,m}y=x+y$, but if $x+y>n$, then $x+_{n,m}y=(x+y)_m\in C_m$; if $x\in H_n$ and $y\in C_m$, then $x+_{n,m}y=x_m+y\in C_m$, and if $x\in C_m$ and $y\in H_n$, then $x+_{n,m}y=x+y_m\in C_m$; finally, if x, $y\in C_m$, then $x+_{n,m}y=x+y$. It is easy to verify that $+_{n,m}$ is a commutative associative composition on $D(n, m)$. Let $D^*(n, m)=H_n^*\cup C_m$. $D^*(n, m)$ is clearly a subsemigroup of $D(n, m)$, obtained by deleting the identity element 0 from $D(n, m)$. Semigroups $D(n-1, m)$ and $D^*(n, m)$ both have $n+m$ elements but are not isomorphic, for the latter has only one element x satisfying $x+x=x$, namely 0_m, whereas the former has two, namely 0 and 0_m. We shall say a semigroup D is a *dipper semigroup* if it is isomorphic either with $D(n, m)$ or $D^*(n, m)$ for some $n\geqq 0$, $m>0$. Dipper semigroups have been discussed in [2] and [3]. The terminology is suggested by the fact that the Big Dipper is a natural model for both $D^*(3, 4)$ and $D(2, 4)$. We imagine the cup of the Big Dipper in an upright position and the handle pointing to the left. Starting from its end, we label the three stars of the handle 1, 2, 3, and proceed-

† Research supported by the National Science Foundation.

ing clockwise around the cup, we identify its four stars with 4_4, 5_4, 6_4, 7_4, 4_4 being the star joining the cup to the handle. If we consider the sequence of elements 1, 1+1, 1+1+1, *etc.*, we first trace out the handle and then encircle the cup infinitely many times in a clockwise fashion. Similarly the Big Dipper is a model for $D(2, 4)$ where we identify the stars of its handle with 0, 1, 2, and the stars of its cup with 3_4, 4_4, 5_4, and 6_4. In an analogous fashion, we may picture a dipper constellation with n stars in the handle and m in the cup as the set of elements for both semigroups $D^*(n, m)$ and $D(n-1, m)$. We note that $D^*(0, m)$ (a dipper with no handle) is simply the cyclic group of integers modulo m, whereas $D^*(n, 1)$ [respectively, $D(n-1, 1)$] (a dipper with no cup to speak of) is isomorphic with the semigroup of integers k satisfying $1 \leqq k \leqq n+1$ [respectively, $0 \leqq k \leqq n$], where composition $\circ$ is defined by $x \circ y = \min\{x+y, n+1\}$ [respectively, $\min\{x+y, n\}$]. $D^*(n, m)$ clearly satisfies (PMI), for we may choose $\beta = 1$ if $n \geqq 1$ and $\beta = 1_m$ if $n = 0$. $D(n, m)$ does not satisfy (PMI) but does satisfy (PMI)′, for we may choose $\alpha = 0$ and $\beta = 1$ if $n \geqq 1$, $\beta = 1_m$ if $n = 0$.

THEOREM 1. *Let D be a semigroup satisfying (PMI). Then D is isomorphic either with the positive integers N^* or with a dipper semigroup $D^*(n, m)$. In particular, D is commutative.*

Proof. We define inductively $n \cdot \beta$ for each positive integer n by $1 \cdot \beta = \beta$, $(k+1) \cdot \beta = k \cdot \beta + \beta$. Clearly the function $n \to n \cdot \beta$ is a homomorphism from N^* into D, and it is onto D by (PMI). Consequently D is commutative. If the homomorphism is an isomorphism, then N^* and D are isomorphic. In the contrary case, there is a smallest natural number n such that $(n+1) \cdot \beta = s \cdot \beta$ for some $s > n+1$; let m be the smallest positive integer such that $(n+1) \cdot \beta = (n+m+1) \cdot \beta$. For any $k \geqq n+1$ and any natural number s it is easy to see by induction that $(k+sm) \cdot \beta = k \cdot \beta$; indeed, if $k = n+1+r$ and $(k+(s-1)m) \cdot \beta = k \cdot \beta$, then $(k+sm) \cdot \beta = (n+1+r+m+(s-1)m) \cdot \beta = (n+1+m) \cdot \beta + (r+(s-1)m) \cdot \beta = (n+1) \cdot \beta + (r+(s-1)m) \cdot \beta = (n+r+1+(s-1)m) \cdot \beta = (k+(s-1)m) \cdot \beta = k \cdot \beta$. Further, if $n+1 \leqq r < s \leqq n+m$, then $r \cdot \beta \neq s \cdot \beta$, for if $r \cdot \beta = s \cdot \beta$, then $(n+r+m-s+1) \cdot \beta = r \cdot \beta + (n+m+1-s) \cdot \beta = s \cdot \beta + (n+m+1-s) \cdot \beta = (n+m+1) \cdot \beta = (n+1) \cdot \beta$ and $0 < r+m-s < m$, contradicting the definition of m. Let $f(k \cdot \beta) = k$ if $1 \leqq k \leqq n$, and $f(k \cdot \beta) = k_m$ if $n+1 \leqq k \leqq n+m$. From the preceding, one easily establishes that f is a well-defined isomorphism from D onto $D^*(n, m)$.

A problem equivalent to finding all semigroups satisfying (PMI) is that of finding all systems $(D, ')$ where D is a set, $'$ a function from D into D satisfying the property (Ind): *There exists $\beta \in D$ such that for any subset S of D, if $\beta \in S$ and if $x' \in S$ whenever $x \in S$, then $S = D$.* (The author is indebted to the referee for this observation.) Indeed, if D satisfies (PMI), the function $'$ defined by $x' = x+\beta$ satisfies (Ind) and also the property $x+y' = (x+y)'$ for all $x, y \in D$. On the other hand, if $'$ is a function satisfying (Ind), there is a unique composition $+$ on D such that (i) $x+\beta = x'$ for all $x \in D$, and (ii) $x+y' = (x+y)'$ for all $x, y \in D$, and further D with this composition is a semigroup satisfying (PMI);

this is proved in [1, pp. 4–6] by an argument (depending only on (Ind) and not on the other Peano postulates) ascribed to Kalmár. Consequently by Theorem 1, every system $(D, ')$ satisfying (Ind) is isomorphic to N^* or some $D^*(n, m)$ in the sense that either there exists a one-one function f from D onto N^* or $D^*(n, m)$ for some $n>0$, $m>0$ satisfying $f(x')=f(x)+1$ for all $x\in D$, or else there exists a one-one function f from D onto $D^*(0, m)=C_m$ for some $m>0$ satisfying $f(x')=f(x)+1_m$ for all $x\in D$. Alternatively, one might first prove that a system $(D, ')$ satisfying (Ind) is isomorphic with N^* or $D^*(n, m)$ for some $n\geqq 0$, $m>0$; from this follows Theorem 1 by the uniqueness of the composition $+$ satisfying (i) and (ii).

3. Almost cyclic semigroups. A particularly simple type of semigroup is one which contains in addition to a subgroup just one element. We first inquire into the possibilities that may occur (in the following theorem we violate the convention that a composition denoted by "$+$" is always commutative).

THEOREM 2. *Let D be a (not necessarily commutative) semigroup with composition $+$ which contains a group G and an element α not in G such that $D=G\cup\{\alpha\}$. Then one and only one of the following holds*: (1) $\alpha+x=x=x+\alpha$ *for all* $x\in D$; (2) $\alpha+x=\alpha=x+\alpha$ *for all* $x\in D$; (3) *there exists* $b\in G$ *such that* $\alpha+x=b+x$ *and* $x+\alpha=x+b$ *for all* $x\in G$ *and* $\alpha+\alpha=b+b$; (4) *G has only one element β, and either* (4a) $\alpha+\beta=\beta+\alpha=\alpha$, $\alpha+\alpha=\beta$, *i.e., D is isomorphic with the additive group of integers modulo* 2, *or* (4b) $\alpha+\alpha=\alpha+\beta=\alpha$, $\beta+\alpha=\beta$, *or* (4c) $\alpha+\alpha=\beta+\alpha=\alpha$, $\alpha+\beta=\beta$. *Conversely if G is a group and if $D=G\cup\{\alpha\}$ where α is not in G, then the composition $+$ of G may be extended to an associative composition on D by defining $\alpha+x$ and $x+\alpha$ to satisfy the identities of* (1), (2), (3), *or, if G has only one element β,* (4a), (4b), *or* (4c).

Proof. We first assert that either $\alpha+x=\alpha$ for all $x\in G$, or else $\alpha+x\in G$ for all $x\in G$. For suppose $\alpha+c\in G$ for some $c\in G$ but $\alpha+d=\alpha$ for some $d\in G$. Then $\alpha+0=\alpha+(c-c)=(\alpha+c)-c\in G$, but $(\alpha+0)+d=\alpha+(0+d)=\alpha+d=\alpha$, whence $\alpha+0$ cannot belong to G, a contradiction. Similarly, either $x+\alpha=\alpha$ for all $x\in G$, or else $x+\alpha\in G$ for all $x\in G$. We next assert that if G has more than one element, either $\alpha+x=\alpha=x+\alpha$ for all $x\in G$, or else $\alpha+x$, $x+\alpha\in G$ for all $x\in G$. Indeed, let $z\neq 0$ be in G. If $\alpha+x=\alpha$ for all $x\in G$ but $x+\alpha\in G$ for all $x\in G$, then $(z+\alpha)+z=z+(\alpha+z)=z+\alpha\in G$ whence $z=0$, a contradiction; similarly it is not possible that $\alpha+x\in G$ for all $x\in G$ but $x+\alpha=\alpha$ for all $x\in G$. Hence by the preceding, the assertion is established. We now consider four cases. (i) $\alpha+\alpha=\alpha$ and either $\alpha+0\in G$ or $0+\alpha\in G$. If $\alpha+0\in G$, then for any $x\in G$, $\alpha+x=y\in G$, whence $y=\alpha+x=(\alpha+\alpha)+x=\alpha+(\alpha+x)=\alpha+y=\alpha+(0+y)=\alpha+((x-x)+y)=(\alpha+x)+(-x+y)=y-x+y$, and therefore $0=-x+y$, *i.e.*, $x=y=\alpha+x$. Similarly if $0+\alpha\in G$, then $x+\alpha=x$ for all $x\in G$. Hence by our previous assertion, if G has more than one element or if D is commutative, (1) holds. In the contrary case, clearly either (4b) or (4c) holds. (ii) $\alpha+\alpha=\alpha$ and $\alpha+0=\alpha=0+\alpha$. Then by a preceding assertion, as $0\in G$, $\alpha+x=\alpha=x+\alpha$ for all $x\in D$, *i.e.*, (2) holds. (iii) $\alpha+\alpha\in G$ and either $\alpha+0\in G$ or $0+\alpha\in G$. If $\alpha+0\in G$,

let $b=\alpha+0$. Then for all $x\in G$, $\alpha+x=\alpha+(0+x)=(\alpha+0)+x=b+x$, and $\alpha+\alpha=(\alpha+\alpha)+0=\alpha+(\alpha+0)=\alpha+b=\alpha+(0+b)=(\alpha+0)+b=b+b$. Similarly, if $0+\alpha\in G$ and if $c=0+\alpha$, then $x+\alpha=x+c$ for all $x\in G$ and $\alpha+\alpha=c+c$. Suppose both $\alpha+0$, $0+\alpha\in G$. Then $b=b+(b+(-b))=b+(\alpha+(-b))=(b+\alpha)+(-b)=b+c+(-b)$, whence $0=c-b$, *i.e.*, $b=c$. Thus if G has more than one element or if D is commutative, (3) holds. But if G has only one element β, then $\alpha+\alpha=\beta$ so $\alpha+\beta=\alpha+(\alpha+\alpha)=(\alpha+\alpha)+\alpha=\beta+\alpha$, so D is indeed commutative. Hence (3) holds. (iv) $\alpha+\alpha\in G$ and $\alpha+0=\alpha=0+\alpha$. Let $c=\alpha+\alpha$. By the preceding, $\alpha+x=\alpha=x+\alpha$ for all $x\in G$. Hence for any $x\in G$, $c=\alpha+\alpha=\alpha+(\alpha+x)=(\alpha+\alpha)+x=c+x$, whence $x=0$. Thus G has only one element 0, and so $\alpha+\alpha=0$, $\alpha+0=\alpha=0+\alpha$, *i.e.*, (4a) holds. Clearly (1), (2), (3), (4a), (4b), (4c) are mutually exclusive. The verification of the remaining assertion of the theorem is easy.

For each positive integer m and each integer n satisfying $0\leqq n\leqq m-1$, let $G(n, m)$ be the semigroup $C_m\cup\{\alpha\}$ where C_m is the additive group of integers modulo m, α is not a member of C_m, and where $\alpha+x=x+\alpha=n_m+x$ for all $x\in C_m$, and $\alpha+\alpha=n_m+n_m$. By Theorem 2, $G(n, m)$ is a commutative semigroup. $G(n, m)$ satisfies (PMI)′, for we may choose α to be the element adjoined to C_m, β to be 1_m. We shall call a semigroup D *almost cyclic* if it is isomorphic to some $G(n, m)$. The function taking α onto 1 and each member of C_m onto itself is an isomorphism of $G(1, m)$ with $D^*(1, m)$. It is easy to see that $G(0, 1)$, which is isomorphic with $D^*(1, 1)$, is the only other almost cyclic semigroup which is a dipper.

THEOREM 3. *Let D be a commutative semigroup satisfying (PMI)′. Then either D is isomorphic with the positive integers N^* or with the natural numbers N, or is a dipper semigroup, or is an almost cyclic semigroup.*

Proof. Let α and β be such that the only subset of D which contains α and contains $x+\beta$ whenever it contains x is D itself. Let E be the subset of all elements of the form $n\cdot\beta$ for positive integers n. We assert that $D=E\cup\{\alpha\}$; by (PMI)′ it suffices to show $\alpha+\beta\in E$. At any rate, by (PMI)′ D is the set of all elements of the form $\alpha+n\cdot\beta$ for $n\geqq 0$. In particular, there exist nonnegative integers k and n such that $\alpha+\alpha=\alpha+k\cdot\beta$ and $\beta=\alpha+n\cdot\beta$. Hence $\alpha+\beta=\alpha+(\alpha+n\cdot\beta)=(\alpha+\alpha)+n\cdot\beta=(\alpha+k\cdot\beta)+n\cdot\beta=(\alpha+n\cdot\beta)+k\cdot\beta=\beta+k\cdot\beta=(k+1)\cdot\beta\in E$. Hence $D=E\cup\{\alpha\}$. Clearly E is a semigroup satisfying (PMI) and so by Theorem 1 is isomorphic either with N^* or with $D^*(n, m)$ for some $n\geqq 0$, $m>0$. Hence we may assume α is not a member of E, for otherwise the proof is complete. If $\alpha+\beta=\alpha$, then $\{\alpha\}$ contains α and contains $x+\beta$ whenever it contains x, so $D=\{\alpha\}$ and thus $\alpha=\beta$, contradicting our assumption α is not in E. Hence $\alpha+\beta\neq\alpha$. Let p be the smallest positive integer such that $\alpha+\beta=p\cdot\beta$. We consider two cases. (1) $p=1$ and $\alpha+\alpha=\alpha$. Then by induction $\alpha+k\cdot\beta=k\cdot\beta$ for all $k\geqq 1$. Therefore D is isomorphic with N if E is isomorphic with N^*, and D is isomorphic with $D(n, m)$ if E is isomorphic with $D^*(n, m)$. (2) Either $\alpha+\alpha=r\cdot\beta$ for some $r\geqq 1$ and $p=1$, or else $p>1$. In the first case,

$(r+1)\cdot\beta=r\cdot\beta+\beta=(\alpha+\alpha)+\beta=\alpha+(\alpha+\beta)=\alpha+\beta=\beta$. In the second case, by (PMI)' D consists of the elements α and $k\cdot\beta$ for $k\geqq p$, so in particular there exists $k\geqq p$ such that $\beta=k\cdot\beta$. Thus in either case there exists $k>1$ such that $\beta=k\cdot\beta$. Let m be the smallest positive integer such that $\beta=(m+1)\cdot\beta$. Then as in Theorem 1, we may conclude that the function f defined by $f(k\cdot\beta)=k_m$, $1\leqq k\leqq m$, is a well-defined isomorphism from E onto the additive group of integers modulo m. By Theorem 2, one of the following possibilities occurs: (1) $\alpha+x=x$ for all $x\in D$; then D is isomorphic with $D(0, m)$. (2) $\alpha+x=\alpha$ for all $x\in D$; but this contradicts $\alpha+\beta\neq\alpha$. (3) There exists $b\in E$ such that $\alpha+x=b+x$ for all $x\in E$ and $\alpha+\alpha=b+b$. Then if the isomorphism f of E onto C_m carries b into n_m, D is clearly isomorphic with $G(n, m)$. (4) D is isomorphic with the integers modulo 2 under addition, the dipper semigroup $D^*(1, 1)$. This completes the proof.

References

1. E. Landau, Grundlagen der Analysis, 2nd American ed., New York, 1948.
2. H. Lenz, Zur Axiomatik der Zahlen, Acta Math. Acad. Sci. Hungar., vol. 9, 1958, pp. 33–44.
3. H. S. Vandiver, Note on a simple type of algebra in which the cancellation law of addition does not hold, Bull. Amer. Math. Soc., vol. 40, 1934, pp. 914–920.

A TOPOLOGY FOR SEQUENCES OF INTEGERS II

R. L. DUNCAN, Pennsylvania State University

1. Introduction. Let S denote the set of all increasing sequences of positive integers which have asymptotic density and define $\rho(A, B)=0$ if $A=B$ and $\rho(A, B)=k^{-1}+|\delta(A)-\delta(B)|$ otherwise, where $\delta(A)$ denotes the asymptotic density of A and where A and B differ in the kth term but do not differ in any previous term. Then S is a totally disconnected nowhere locally compact separable metric space [1]. This metric for S seems to be a very natural one, although most of the simple topological properties of S do not depend on the fact that $\delta(A)$ represents the asymptotic density of A. The most characteristic properties of $\delta(A)$ in this respect appear to be that $\delta(A)=\delta(B)$ if A and B differ in only a finite number of terms and that the set of values of $\delta(A)$ is an interval.

It would of course be of some interest to determine properties which completely characterize $\delta(A)$, *i.e.*, to provide an axiomatic approach to density. We believe that a certain multiplicative property due to Niven [1, 2, 3] may be of considerable importance in this connection. In the present note we continue the study of the properties of S begun in Part I.

2. Decomposition of S. It has been shown previously [1] that if $S=E_1\cup E_2$ then $\rho(E_1, E_2)=0$ or n^{-1} where n is a positive integer, and that the second possibility actually occurs for every n. We shall derive a necessary condition for

$\rho(E_1, E_2)>0$. The following notation will be used for this purpose. If $E\subset S$, let E^* denote the set of all real numbers α such that $\alpha=\delta(A)$ for some $A\in E$. Also, let I denote the open interval (0, 1).

THEOREM 1. *If $E\subset S$, then either $\rho(E, S-E)=0$ or $E^*=\overline{I}$.*

Proof. Assume that $\rho(E, S-E)>0$. Then $S-E$ is closed. If E^* contains a limit point of $\overline{I}-E^*$, then E contains a limit point of $S-E$. Hence $\overline{I}-E^*$ is closed and $E^*\cap I$ is open. By the component interval theorem [4] $E^*\cap I$ is a countable union of mutually disjoint open intervals. If $E^*\cap I\neq I$, then there exists a $t\in I$ which is an end point of one of these intervals. Now let $T=\{A\in S\,|\,\delta(A)=t\}$. Then there exists $\{A_n\}\subset E$ and $\{B_n\}\subset T\subset(S-E)$ such that $\rho(A_n, B_n)\to 0$. But this implies that $\rho(E, S-E)=0$ and so $E^*\cap I=I$. Thus $I\subset E^*$ and the same argument applied to $t=0$ and $t=1$ shows that $E^*=\overline{I}$.

COROLLARY. *If $S=E_1\cup E_2$ and if $\rho(E_1, E_2)>0$, then $E_1^*=E_2^*=\overline{I}$.*

3. Category of S. We recall that a set is of the first category (in itself) if it is a countable union of nowhere-dense sets.

THEOREM 2. *S is a set of the first category.*

Proof. Let $A=\{a_n\}$ and let R denote the set of all real numbers of the form $a=\sum_{r=1}^{\infty}2^{-a_r}$ with $A\in S$. Then $\pi: A\to a$ is a continuous one-one mapping of S on R. It follows from results due to Hill [5] that R is a set of the first category in (0, 1]. Thus $S=\bigcup_{n=1}^{\infty}E_n$ where $\pi(E_n)$ is nowhere-dense in (0, 1]. Suppose that $\overline{E}_n$ contains a sphere $S(A, r)=\{B\in S\,|\,\rho(A, B)<r\}$. Now $S(A, r)=\bigcup_{k>r^{-1}}\{S_k(A)\cap S_k(A, r)\}$ where $S_k(A)$ is the set of all $B\in S$ which agree with A in the first $k-1$ or more terms and $S_k(A, r)=\{B\in S\,|\,|\delta(A)-\delta(B)|<r-k^{-1}\}$. For convenience we may take $r=m^{-1}$ for some sufficiently large positive integer m. Then it is clear that $\pi\{S(A, r)\}$ is everywhere-dense in the interval $(x, x+2^{-a_{m+1}})$, where $x=\sum_{r=1}^{m+1}2^{-a_r}$. Since $\pi(\overline{E}_n)\subset\overline{\pi(E_n)}=\overline{\overline{\pi(E_n)}}$, it follows that $\overline{\pi(E_n)}$ contains an interval. But this is impossible and so E_n is nowhere-dense in S and S is a set of the first category.

4. A related space. Let T denote the set of all increasing sequences (finite or infinite) of positive integers with the metric $\rho(A, B)=0$ if $A=B$ and $\rho(A, B)=k^{-1}$ otherwise, where k has the same meaning as before. This type of metric has been discussed by Ostmann [6].

THEOREM 3. *T is homeomorphic to the set of positive elements of the Cantor set.*

Proof. Let $A=\{a_n\}$ and $x=2\sum_{a_r\in A}3^{-a_r}$ and consider the mapping $\tau: A\to x$. Since none of the digits in the ternary expansion of x are one's, the mapping is one-one. Also, several simple calculations show that $3^{-a_{k_n}}\leqq|x_n-x|\leqq 3\cdot 3^{-a_{k_n}}$ if $x_n\neq x$ and where $\rho(A_n, A)=k_n^{-1}$. Thus $x_n\to x$ if and only if $k_n\to\infty$, or if all but a finite number of the A_n's are equal to A, *i.e.*, if $A_n\to A$, and τ is a homeomorphism.

Let T^* denote the space of all nonempty subsets of the set of all positive integers where the limit of a sequence of sets $\{E_n\}$ is defined by $\lim_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} (\bigcap_{k=n}^{\infty} E_k)$. It is known [7, p. 145] that T^* is homeomorphic to the set of positive elements of the Cantor set under the mapping τ. We may therefore conclude that T and T^* are homeomorphic under the identity mapping.

It can be shown by means of results such as Theorem 3 that S is homeomorphic to a certain subset of the plane. Let C denote the set of elements $x = \sum_{n=1}^{\infty} \epsilon_n(x)3^{-n}$ of the Cantor set for which $\phi(x) = \lim_{n\to\infty} (1/2n) \sum_{r=1}^{n} \epsilon_r(x)$ exists and where $\epsilon_n(x) = 0$ or 2 with the 2 occurring an infinite number of times. Then $\phi(x)$ is a real-valued function defined on C and, since the ternary representation of any $x \in C$ is unique, $\phi(x)$ is also single-valued. The following result is now obvious.

THEOREM 4. *S is homeomorphic to the graph of the function $\phi(x)$.*

References

1. R. L. Duncan, A topology for sequences of integers I, this MONTHLY, vol. 66, 1959, pp. 34–39.
2. R. L. Duncan, Problem 4838, this MONTHLY, vol. 66, 1959, p. 238.
3. Ivan Niven, The asymptotic density of sequences, Bull. Amer. Math. Soc., vol. 57, 1951, pp. 420–434.
4. I. P. Natanson, Theory of Functions of a Real Variable, New York, 1955.
5. J. D. Hill, Summability of sequences of 0's and 1's, Ann. of Math., vol. 46, 1945, pp. 556–562.
6. H. H. Ostmann, Additive Zahlentheorie, 2 vols., Berlin, 1956.
7. W. Sierpinski, General Topology, University of Toronto Press, 1952.

SOME REMARKS ON ARITHMETICAL IDENTITIES*

P. J. McCARTHY, Florida State University

1. Introduction. A single-valued function $f(n, r)$, where n is a nonnegative integer and r is a positive integer, having values in the field of complex numbers, will be said to belong to the class E_k if for all n and r, $f(n, r) = f((n, r^k)_k, r)$; here $(a, b)_k$ is the largest integral kth power which divides both a and b. This class of functions was studied in [9] (it was defined in a slightly different manner there) and it was shown there that if $f(n, r)$ belongs to the class E_k then

$$f(n, r) = \sum_{d|r} \alpha(d, r)c_k(n, d),$$

where $c_k(n, r)$ is the extension of Ramanujan's sum $c(n, r)$ defined by Cohen in [2], and the "Fourier coefficients" $\alpha(d, r)$ are uniquely determined by $f(n, r)$.

* This work was supported by a Summer Research Award from the Research Council of the Florida State University.

This extended a result of Cohen, who had previously studied the class E_1 [4]. Using his results for the class E_1 Cohen developed in [6] and [7] a systematic procedure for the study of arithmetical identities involving the classical and some new arithmetic functions. In a previous paper [9] we have shown that Cohen's methods, when applied to the class E_k for arbitrary positive integral k, yield much more general identities, including many of the classical identities in their full generality. This paper is a continuation of [9] in which we shall apply the methods used by Cohen in [7] to the class E_k.

In Sections 2, 3, 4 and 5 we make use of an orthogonality property of the functions $c_k(n, r)$ under the Cauchy product to study the Cauchy products of functions of the class E_k. This leads to generalizations of all the identities of [7], but we shall give only a few examples to illustrate how these generalizations may be obtained. In particular, in Section 4, we obtain a generalization of an identity of Brauer and Rademacher [1]. In Section 6 we study the number of solutions of a certain congruence, and in Section 7 we discuss generalizations of certain Busche-Ramanujan identities.

Two functions which we shall have occasion to use often are defined as follows: for an arbitrary complex number s,

$$\phi_s(r) = \sum_{d^s \mid r} d^s \mu(e), \qquad \sigma_s(r) = \sum_{d \mid r} d^s.$$

When s is a positive integer, $\phi_s(r) = J_s(r)$, Jordan's extension of Euler's totient function. We shall also make use of the fact that [2, Theorem 2]

$$c_k(n, r) = \sum_{d^k \mid (n, r^k)_k} d^k \mu(r/d).$$

Then $c_k(0, r) = \phi_k(r)$. For a discussion of the function $\phi_k(r)$ (k a positive integer) the reader is referred to [5].

2. The Cauchy product. Let $f(n, r)$ and $g(n, r)$ belong to the class E_k. Then [9, Theorem 1]

$$f(n, r) = \sum_{d \mid r} \alpha(d, r) c_k(n, d), \tag{1}$$

$$g(n, r) = \sum_{d \mid r} \beta(d, r) c_k(n, d), \tag{2}$$

where $\alpha(d, r)$ and $\beta(d, r)$ are uniquely determined by $f(n, r)$ and $g(n, r)$, respectively. The function $c_k(n, r)$ has the following orthogonality property [3, Theorem 1]: if $d \mid r$ and $e \mid r$ then

$$\sum_{a+b \pmod{r^k}} c_k(a, d) c_k(b, e) = \begin{cases} r^k c_k(n, d) & \text{if } d = e, \\ 0 & \text{if } d \neq e. \end{cases}$$

Using this it is easily seen that

THEOREM 1. $\sum_{a+b \pmod{r^k}} f(a, r) g(b, r) = r^k \sum_{d \mid r} \alpha(d, r) \beta(d, r) c_k(n, d)$.

Let $g(r)$ be a function of a single positive integral variable, and set

$$G_s(r) = \sum_{d|r} d^s g(r/d), \qquad \Gamma_k^{(s)}(n, r) = \sum_{d^k | (n, r^k)_k} d^s g(r/d),$$

where s is a complex number. We have shown in [9] that

$$\Gamma_k^{(ks)}(n, r) = \sum_{d=r} d^{k(s-1)} G_{k(s-1)}(e) c_k(n, d).$$

Starting with a second function $h(r)$ we define $H_s(r)$ and $\Delta_k^{(s)}(n, r)$ in the same way as we defined $G_s(r)$ and $\Gamma_k^{(s)}(n, r)$, respectively. If we apply Theorem 1 to $\Gamma_k^{(ks)}(n, r)$ and $\Delta_k^{(kt)}(n, r)$ we obtain

$$(3) \qquad \sum_{n \equiv a+b \pmod{r^k}} \Gamma_k^{(ks)}(a, r) \Delta_k^{(kt)}(b, r) = r^k \sum_{d=r} d^{k(s+t-2)} G_{k(s-1)}(e) H_{k(t-1)}(e) c_k(n, d).$$

Example 1. Let $g(r) = 1$ for all r. Then $G_s(r) = \sigma_s(r)$ and $\Gamma_k^{(s)}(n, r) = \sigma_s((n, r^k)_k^{1/k})$, which we denote by $\sigma_{s,k}(n, r)$. If we also let $h(r) = 1$ for all r we have from (3), replacing ks by s and kt by t,

$$\sum_{n \equiv a+b \pmod{r^k}} \sigma_{s,k}(a, r) \sigma_{t,k}(b, r) = r^k \sum_{d=r} d^{s+t-2k} \sigma_{s-k}(e) \sigma_{t-k}(e) c_k(n, d)$$

(see [7, Corollary 1]).

Example 2. Let $Q_k(n, r, s)$ be the number of solutions $x_i \pmod r$, $y_i \pmod{r^k}$, $i = 1, 2, \cdots, s$, of the congruence

$$n \equiv a_1 x_1^k y_1 + \cdots + a_s x_s^k y_s \pmod{r^k},$$

where $(a_i, r) = 1$ for all i. Cohen has shown [3, Theorem 5] that

$$Q_k(n, r, s) = r^{k(s-1)+s} \sum_{d|r} \frac{c_k(n, d)}{d^s}.$$

If we set $g(r) = r^{k(s-1)} \phi_s(r)$ then [9, Example 3] $G_{k(s-1)}(r) = r^{k(s-1)+s}$ and $\Gamma_k^{(ks)}(n, r) = Q_k(n, r, s)$. Hence from (3), with kt replaced by t,

$$\sum_{n \equiv a+b \pmod{r^k}} Q_k(a, r, s) \sigma_{t,k}(b, r) = r^{k(s-1)} \sum_{d=r} d^{t-s-k} \sigma_{t-k}(e) c_k(n, d).$$

Example 3. Let $\phi_k^{(s)}(n, r)$ be the number of solutions of the linear congruence

$$n \equiv x_1 + \cdots + x_s \pmod{r^k},$$

where $(x_i, r^k)_k = 1$ for $i = 1, 2, \cdots, s$. Then [9, Example 6],

$$\phi_k^{(s)}(n, r) = \frac{\phi_k^s(r)}{r^{ks}} \sum_{d|r} \frac{\mu^s(d)}{\phi_k^s(d)} c_k(n, d).$$

We then have

$$(4)\qquad \sum_{n\equiv a+b \pmod{r^k}} \sigma_{s,k}(a, r)\phi_k^{(t)}(b, r) = \phi_k^t(r) \sum_{d\mid r} \frac{d^{s-k}\sigma_{s-k}(e)\mu^t(d)}{\phi_k^t(d)}\, c_k(n, d)$$

(see [7, Corollary 4]). Let $\tau_k(r)$ be the number of divisors of r whose kth power also divides r, and set $\tau_k(n, r)=\tau_k((n, r^k)_k)$. Then $\tau_k(n, r)=\sigma_{0,k}(n, r)$. If we set $s=0$ and $t=2$ in (4) we obtain

$$\sum_{n\equiv a+b \pmod{r^k}} \tau_k(a, r)\theta_k(b, r) = \frac{\phi_k^2(r)}{r^k} \sum_{d\mid r} \frac{\sigma_k(e)\,|\,\mu(d)\,|}{\phi_k^2(d)}\, c_k(n, d),$$

where $\theta_k(n, r)$ generalizes Nagell's totient function [9, Example 6]. As an identity connecting the number of solutions of the linear and semilinear congruences we have

$$\sum_{n\equiv a+b \pmod{r^k}} Q_k(a, r, s)\phi_k^{(t)}(b, r) = r^{k(s-1)+s} \sum_{d\mid r} \frac{\mu^t(d)}{d^s\phi_k^t(d)}\, c_k(n, d).$$

3. The case $n=0$. If $f(n, r)$ and $g(n, r)$ are in the class E_k and are given by (1) and (2), respectively, then

THEOREM 2. $\sum_{d\mid r} f(e^k, r)g(e^k, r)\phi_k(d)=r^k\sum_{d\mid r}\alpha(d, r)\beta(d, r)c_k(n, d)$.

The proof is similar to that of [7, Theorem 5]. It follows immediately from this theorem that

$$\sum_{d\mid r} \Gamma_k^{(s)}(e^k, r)\Delta_k^{(t)}(e^k, r)\phi_k(d) = r^k \sum_{d\mid r} d^{s+t-2k} G_{s-k}(e)H_{t-k}(e)\phi_k(d).$$

Example 4. If we set $g(r)=1$ for all r then $G_s(r)=\sigma_s(r)$ and if $e\mid r$, $\Gamma_k^{(s)}(e^k, r)=\sigma_s(e)$. Hence

$$\sum_{d\mid r} \sigma_s(e)\sigma_t(e)\phi_k(d) = r^k \sum_{d\mid r} d^{s+t-2k}\sigma_{s-k}(e)\sigma_{t-k}(e)\phi_k(d)$$

(see [7, Corollary 29]). When $s=t=k$ this yields

$$\sum_{d\mid r} \sigma_k^2(e)\phi_k(d) = r^k \sum_{d\mid r} \tau^2(e)\phi_k(d).$$

Example 5. When $g(r)=\mu(r)$ for all r, $G_s(r)=\phi_s(r)$ and if $e\mid r$, $\Gamma_k^{(s)}(e^k, r)=\phi_s(r)\mu(r/e)/\phi_s(r/e)$ [9, Example 5]. Hence

$$\phi_s(r)\phi_t(r) \sum_{d\mid r} \frac{|\,\mu(d)\,|\,\phi_k(d)}{\phi_s(d)\phi_t(d)} = r^k \sum_{d\mid r} d^{s+t-2k}\phi_{s-k}(e)\phi_{t-k}(e)\phi_k(d)$$

(compare this, when $s=t$, with [7, Corollary 26]). If we combine the functions used in this example and in the last we obtain

$$\phi_s(r)\sum_{d\,e=r}\frac{\mu(d)\phi_k(d)\sigma_t(e)}{\phi_s(d)}=r^k\sum_{d\,e=r}d^{s+t-2k}\phi_{s-k}(e)\sigma_{t-k}(e)\phi_k(d).$$

4. Generalization of the Brauer-Rademacher identity. In [7, Section 5] Cohen gave a proof of an interesting identity due to Brauer and Rademacher [1]. If we apply the method of his proof to functions of the class E_k we can generalize this identity. We have

$$\begin{aligned}\sum_{(b,r^k)_k=1}c_k(n-b,r)&=\sum_{(b,r^k)_k=1}\sum_{d^k\mid(n-b,r^k)_k}d^k\mu(r/d)\\&=\sum_{(b,r^k)_k=1}\sum_{\substack{d\mid r\\ n\equiv b\,(\mathrm{mod}\,d^k)}}d^k\mu(r/d)\\&=\sum_{d\mid r}d^k\mu(r/d)\sum_{(b,r^k)_k=1}1\\&=\phi_k(r)\sum_{\substack{d\mid r\\(n,d^k)_k=1}}\frac{d^k}{\phi_k(d)}\mu(r/d),\end{aligned}$$

where we have used the fact that if $(n,d^k)_k=1$, where $d\mid r$, then there are exactly $\phi_k(r)/\phi_k(d)$ k-reduced residues b (mod r^k) congruent to n (mod r^k).

Now choose a function $g(r)$, define $G_s(r)$ and $\Gamma_k^{(s)}(n,r)$ as in Section 2, and set $G_s^*(n,r)=m^sG_s(r/m)$ where $m^k=r^k/(n,r^k)_k$. Then we have shown [9, Section 4] that

$$G_{ks}^*(n,r)=r^{-k}\sum_{d\,e=r}\Gamma_k^{(k(s+1))}(e^k,r)c_k(n,d).$$

Hence, if $f(n,r)$ is any function of the class E_k and is given by (1), we have by Theorem 1,

$$\sum_{n\equiv a+b\,(\mathrm{mod}\,r^k)}f(a,r)G_{ks}^*(b,r)=\sum_{d\,e=r}\alpha(d,r)\Gamma_k^{(k(s+1))}(e^k,r)c_k(n,d). \tag{5}$$

We are interested in the case when $s=0$ and $g(r)=\mu(r)$ for all r. Then $G_0(r)=1$ or 0 according as $r=1$ or $r\neq 1$ and so $G_0^*(n,r)=1$ or 0 according as $(n,r^k)_k=1$ or $\neq 1$. Furthermore, $\Gamma_k^{(k)}(e^k,r)=\phi_k(r)\mu(r/e)/\phi_k(r/e)$ if $e\mid r$. Hence (5) becomes

$$\sum_{(b,r^k)_k=1}f(n-b,r)=\phi_k(r)\sum_{d\mid r}\frac{\alpha(d,r)\mu(d)}{\phi_k(d)}c_k(n,d) \tag{6}$$

(see [7, (5.4)]). In particular, when $f(n,r)=c_k(n,r)$ we have

$$\phi_k(r)\sum_{\substack{d\mid r\\(n,d^k)_k=1}}\frac{d^k}{\phi_k(d)}\mu(r/d)=\mu(r)c_k(n,r)$$

(see [7, Corollary 34]), and if we use Cohen's evaluation of $c_k(n,r)$ we obtain the desired generalization of the Brauer-Rademacher identity:

$$\phi_k(r) \sum_{\substack{d|r \\ (n,d^k)_k=1}} \frac{d^k}{\phi_k(d)}\mu(r/d) = \mu(r) \sum_{d^k|(n,r^k)_k} d^k\mu(r/d).$$

5. Additional identities. If $f(n, r)$ belongs to the class E_k and has Fourier coefficients $\alpha(d, r)$, we can evaluate $\phi_k^{-1}(r)\sum_{(b,r^k)_k=1} f(n-b, r)$ in two ways (one of which is given by (6)) to obtain the following extension of [7, Theorem 6]:

$$\sum_{\substack{d|r \\ (d,n^k)_k=1}} \frac{d^k}{\phi_k(d)}\left(\sum_{e|(r/d)} \alpha(ed, r)\mu(e)\right) = \sum_{d|r} \frac{\alpha(d, r)\mu(d)}{\phi_k(d)} c_k(n, d).$$

Then, if we choose a function $h(r)$ of a single positive integral variable r, set $g(r) = \sum_{d|r} h(d)$, and consider the function $f(n, r) = \sum_{d|r} g(r/d)c_k(n, d)$, we obtain

$$\sum_{\substack{d|r \\ (d,n^k)_k=1}} \frac{d^k}{\phi_k(d)} h(r/d) = \sum_{d|e|r} \frac{g(e)\mu(d)}{\phi_k(d)} c_k(n, d)$$

(see [7, Theorem 7]).

Example 6. Let $h(r) = r^{k-s}$. Then $g(r) = \sigma_{s-k}(r)/r^{s-k}$ and so we have

$$\sum_{\substack{d|r \\ (d,n^k)_k=1}} \frac{d^s}{\phi_k(d)} = \sum_{d|e|r} \frac{d^{s-k}\sigma_{s-k}(e)\mu(d)}{\phi_k(d)} c_k(n, d)$$

(see [7, Corollary 36]).

Example 7. If we let $h(r) = \mu(r)r^{k-s}$ then $g(r) = r^{k-s}\phi_{s-k}(r)$, and so

$$\sum_{\substack{d|r \\ (d,n^k)_k=1}} \frac{d^s}{\phi_k(d)}\mu(r/d) = \sum_{d|e|r} \frac{d^{s-k}\phi_{s-k}(e)\mu(d)}{\phi_k(d)} c_k(n, d)$$

(see [7, Corollary 37]).

Example 8. Let $h(r) = \phi_s(r)$. Then $g(r) = r^s$ and we have

$$\sum_{\substack{d|r \\ (d,n^k)_k=1}} \frac{d^k}{\phi_k(d)}\phi_s(r/d) = r^s \sum_{d|r} \frac{\mu(d)}{d^s\phi_k(d)} c_k(n, d).$$

When $n = 1$ this yields

$$\sum_{d|e|r} d^k \frac{\phi_s(e)}{\phi_k(d)} = r^s \sum_{d|r} \frac{|\mu(d)|}{d^s\phi_k(d)}.$$

6. A congruence. In this section we obtain a formula for the number of solutions $N_k^{(s)}(n, r)$ of the congruence $n \equiv x_1 + \cdots + x_s \pmod{r^k}$, where $(x_i, r^k)_k$

is a $2k$th power for $i=1, 2, \cdots, s$. Our discussion follows that of [7, Section 3], where Cohen considered the case $k=1$. Let $\psi_k(r)$ be the number of integers $a \pmod{r^k}$ such that $(a, r^k)_k$ is a $2k$th power. Also, let $\nu(r)=1$ or 0 according as r is or is not a square, and $\nu_k(n, r)=\nu((n, r^k)_k^{1/k})$. Finally, let $\lambda(r)=(-1)^{\Omega(r)}$, where $\Omega(r)$ is the number of primes dividing r with repetitions counted.

If we set

$$\beta_s(r) = \sum_{d|r} d^s\lambda(r/d), \qquad L_k^{(s)}(n, r) = \sum_{d^k|(n,r^k)_k} d^s\lambda(r/d),$$

then by the results described in Section 2,

$$L_k^{(s)}(n, r) = \sum_{d^2=r} d^{s-k}\beta_{s-k}(e)c_k(n, d)$$

(see [6, Corollary 3]), which yields when $s=k$,

$$\sum_{d^k|(n,r^k)_k} d^k\lambda(r/d) = \sum_{d^2=r} \nu(e)c_k(n, d) = \sum_{d^2=r} c_k(n, d)$$

(see [6, Corollary 4]). When $n=0$ we obtain

$$\beta_k(r) = \sum_{d^2=r} \phi_k(d).$$

This implies that $\psi_k(r)=\beta_k(r)$ (see [6, Corollary 4.2]). Now applying the identity involving $G_s^*(n, r)$ in Section 4 to the case under consideration here, we obtain

$$\frac{\beta_s((n, r^k)_k^{1/k})}{(n, r^k)_k^{s/k}} = r^{-(s+k)} \sum_{d^2=r} \lambda(d)\beta_{s+k}(e)c_k(n, d)$$

(see [6, Corollary 10]) which yields, when we set $s=0$,

$$\nu_k(n, r) = r^{-k} \sum_{d^2=r} \lambda(d)\beta_k(e)c_k(n, r)$$

(see [6, Corollary 10.4]). We are now able to prove

THEOREM 3. $N_k^{(s)}(n, r)=r^{-k}\sum_{d^2=r}(\lambda(d)\beta_k(e))^s c_k(n, d)$.

Proof. We have just shown that $N_k^{(1)}(n, r)=\nu_k(n, r)$, *i.e.*, that the theorem is true when $s=1$. Now assume that the theorem has been proved for all $t<s$. Then, using Theorem 1,

$$N_k^{(s)}(n, r) = \sum_{n\equiv a+b \pmod{r^k}} N_k^{(s-1)}(a, r)N_k^{(1)}(b, r) = r^{-k} \sum_{d^2=r} (\lambda(d)\beta_k(e))^s c_k(n, d).$$

7. Some Busche-Ramanujan identities. In his development of the theory of multiplicative functions [10], Vaidyanathaswamy considered, for a multiplicative function $f(r)$, identities of the following type:

$$f(MN) = \sum_{d|(M,N)} f(M/d)f(N/d)F(d). \tag{7}$$

He called such an identity a Busche-Ramanujan identity because of the classical identity involving the function $\sigma_k(r)$,

$$\sigma_k(MN) = \sum_{d|(M,N)} d^k \sigma_k(M/d)\sigma_k(N/d)\mu(d). \tag{8}$$

Vaidyanathaswamy obtained a necessary and sufficient condition for a multiplicative function to admit a Busche-Ramanujan identity [**10**, Theorem XXXV]. The identity (8) has been generalized by Cohen [**7**, Theorem 10] and he obtained as special cases of this generalization several interesting arithmetical identities [**7**, Section 6]. However, there is a still more general identity that includes (8) and Cohen's generalization of (8).

THEOREM 4.

$$\sum_{d|(M,N)} d^s \sigma_s(M/d)\sigma_s(N/d)c_k(n, d) = \sum_{\substack{d|(M,N) \\ d^k|n}} d^{s+k}\sigma_s(MN/d^2). \tag{9}$$

Proof. The left-hand side of (9) is equal to

$$\begin{aligned} &\sum_{d|(M,N)} d^s \sigma_s(M/d)\sigma_s(N/d) \sum_{e^k|(n,d^k)_k} e^k \mu(d/e) \\ &= \sum_{\substack{e|(M,N) \\ e^k|n}} e^k \sum_{\substack{d|(M,N) \\ d=\delta e}} d^s \sigma_s(M/d)\sigma_s(N/d)\mu(d/e) \\ &= \sum_{\substack{e|(M,N) \\ e^k|n}} e^{k+s} \sum_{\delta|(M/e,N/e)} \delta^s \sigma_s((M/e)/\delta)\sigma_s((N/e)/\delta)\mu(\delta), \end{aligned}$$

and so, using (8), we obtain the right-hand side of (9).

Note that in (9), k need not be a positive integer. If we set $n=0$ in (9) we obtain

$$\sum_{d|(M,N)} d^s \sigma_s(M/d)\sigma_s(N/d)\phi_k(d) = \sum_{d|(M,N)} d^{s+k}\sigma_s(MN/d^2). \tag{10}$$

If we set $M=N=r$ in (10) we obtain

$$\sum_{de=r} d^s \sigma_s^2(e)\phi_k(d) = \sum_{de=r} d^{s+k}\sigma_s(e^2)$$

which for $s=0$ gives

$$\sum_{de=r} \tau^2(e)\phi_k(d) = \sum_{de=r} d^k \tau(e^2) \tag{11}$$

(see [**7**, Corollaries 45.3 and 45.4]). (11) is an identity due originally to Gegenbauer [**8**, page 298].

A divisor d of an integer r is called a direct divisor of r if $(d, r/d)=1$. We say that a multiplicative function $f(r)$ admits a restricted Busche-Ramanujan identity if (7) holds whenever M and N have no direct divisors in common, *i.e.*, whenever M and N share no prime divisor to the same power. This termi-

nology is also due to Vaidyanathaswamy, and he obtained a necessary and sufficient condition for $f(r)$ to admit such a restricted identity [10, Theorem XXXVIII]. He attributes the fact that Euler's function $\phi(r)$ admits the restricted identity

$$\phi(MN) = \sum_{d|(M,N)} d\phi(M/d)\phi(N/d) \tag{12}$$

to Pillai. We shall now generalize (12). According to Vaidyanathaswamy's characterization of the functions which admit a restricted identity, the function $\phi_s(r)$ admits such an identity since it is a totient function in the sense of [10]. The results of [10] actually give the correct identity for this function, but we shall obtain this identity in such a way that gives us an idea of the "error" in the exceptional cases when the identity does not hold. Note that s need not be a positive integer.

THEOREM 5. *If M and N have no common direct divisors, then*

$$\phi_s(MN) = \sum_{d|(M,N)} d^s\phi_s(M/d)\phi_s(N/d). \tag{13}$$

Proof. We set $f(M, N)$ equal to the right-hand side of (13). If $(M_1N_1, M_2N_2) = 1$ then $((M_1, N_1), (M_2, N_2)) = 1$ and $(M_1M_2, N_1N_2) = (M_1, N_1)(M_2, N_2)$, and therefore $f(M_1M_2, N_1N_2) = f(M_1, N_1)f(M_2, N_2)$. Consequently, in order to evaluate $f(M, N)$ we need only evaluate $f(p^a, p^b)$ where p is a prime. We have, assuming without loss of generality that $a \leqq b$,

$$f(p^a, p^b) = \sum_{i=0}^{a} p^{si}\phi_s(p^{a-i})\phi_s(p^{b-i}),$$

and after some manipulation we obtain

$$f(p^a, p^b) = \begin{cases} \phi_s(p^{a+b}) & \text{if } a \neq b, \\ \phi_s(p^{2a}) + p^{s(a-1)} & \text{if } a = b. \end{cases}$$

This completes the proof of the theorem.

We now set $\sigma_k(n, r) = \sigma_k((n, r))$ and obtain the following extension of (13).

THEOREM 6. *If M and N have no common direct divisors, then*

$$\sum_{d|(M,N)} d^s\phi_s(M/d)\phi_s(N/d)(n, d)^{-k}\sigma_k(n, d) = \sum_{\substack{d|(M,N) \\ d|n}} d^{s-k}\phi_s(MN/d^2). \tag{14}$$

Proof. The left-hand side of (14) is equal to

$$\sum_{d|(M,N)} d^s\phi_s(M/d)\phi_s(N/d) \sum_{e|(n,d)} e^{-k} = \sum_{\substack{e|(M,N) \\ e|n}} e^{-k} \sum_{\substack{d|(M,N) \\ d=\delta e}} d^s\phi_s(M/d)\phi_s(N/d)$$

$$= \sum_{\substack{e|(M,N) \\ e|n}} e^{s-k} \sum_{\delta|(M/e,N/e)} \delta^s\phi_s((M/e)/\delta)\phi_s((N/e)/\delta).$$

Now if M and N have no common direct divisors, then the same is true of M/e and N/e where $e \mid (M, N)$. Hence, using (13), we obtain the right-hand side of (14).

If we set $n=0$ in (14) we obtain

$$\sum_{d \mid (M,N)} d^{s-k}\phi_s(M/d)\phi_s(N/d)\sigma_k(d) = \sum_{d \mid (M,N)} d^{s-k}\phi_s(MN/d^2), \tag{15}$$

which holds whenever M and N have no common direct divisors. If we set $M=r$ and $N=r^2$ then M and N have no common direct divisors and so we have from (15)

$$\sum_{d \mid r} d^{s-k}\phi_s(e)\phi_s(re)\sigma_k(d) = \sum_{d \mid r} d^{s-k}\phi_s(re^2). \tag{16}$$

If we do the same thing in (13) we obtain

$$\sum_{d \mid r} d^s\phi_s(e)\phi_s(re) = \phi_s(r^2).$$

We also have the following remarkable pair of identities: for all integers r and all s,

$$\sum_{d \mid r} d^s\phi_s(e)\phi_s(re)\sigma_{-s}(d) = \sum_{d \mid r} \phi_s(rd^2)$$

and

$$\sum_{d \mid r} d^s\sigma_s(e)\sigma_s(re)\phi_{-s}(d) = \sum_{d \mid r} \sigma_s(rd^2).$$

The first of these identities is obtained from (16) by setting $s=k$ and using the fact that $\sigma_s(r)=r^s\sigma_{-s}(r)$. The second is obtained from (10) by setting $M=r$ and $N=r^2$ and replacing k by $-s$.

References

1. A. Brauer and H. Rademacher, Aufgabe 31, Jber. Deutschen Math.-Verein., vol. 35, 1928, pp. 94–95 (supplement).
2. Eckford Cohen, An extension of Ramanujan's sum, Duke Math. J., vol. 16, 1949, pp. 85–90.
3. ———, An extension of Ramanujan's sum. II. Additive properties, Duke Math. J., vol. 22, 1955, pp. 543–550.
4. ———, A class of arithmetical functions, Proc. Nat. Acad. Sci., U.S.A., vol. 41, 1955, pp. 939–944.
5. ———, Some totient functions, Duke Math. J., vol. 23, 1956, pp. 515–522.
6. ———, Representations of even functions (mod r), I, Arithmetical identities, Duke Math. J., vol. 25, 1958, pp. 401–422.
7. ———, Representations of even functions (mod r), II, Cauchy products, Duke Math. J., vol. 26, 1959, pp. 165–182.
8. L. E. Dickson, History of the Theory of Numbers, vol. I, Washington, 1919.
9. P. J. McCarthy, The generation of arithmetical identities, J. Reine Angew. Math., vol. 203, 1960, pp. 55–63.
10. R. Vaidyanathaswamy, The theory of multiplicative arithmetic functions, Trans. Amer. Math. Soc., vol. 33, 1931, pp. 579–662.

AXIOMS FOR THE AFFINE LINE*

F. CUNNINGHAM, JR., Bryn Mawr College AND CHARLES W. VALENTINE,
Pratt and Whitney Aircraft Company

1. Introduction. The subject of this paper is the structure of a *line*, by which we mean a one-dimensional affine (or Euclidean) space. This structure is less rich than that of the *real line*, defined (somewhat redundantly) as a complete ordered field, but richer than that of the *ordered line*, defined as the real line with all structure other than its order ignored. A precise definition of "line" as mathematical system can be given in terms of the real line: *it is the set of real numbers with just such structure as is invariant under the group of affine functions*: $f(x)=ax+b$, *with* $a\neq 0$. However, in view of the intuitively geometric (rather than arithmetic) nature of the line, it would be more natural to proceed in the opposite direction, namely give an axiomatic definition of *line* and then show its relationship to the real line. Various axiomatic descriptions are certainly possible, based on different primitive concepts such as congruence or translations. The one we are proposing begins with order and prescribes further *how much* structure the line is to have, rather than specifically *what* structure. The quantity of structure is measured by the degree of transitivity of its automorphism group, as will be explained below.

It would be even better if, having given an axiomatic description of the line, we could define from it a model for the real field in a natural way. This is still undone; all we achieve is a proof that the axiomatic description does give the system we intended and no others.

The idea of Klein's Erlanger Program applies not only to geometry but everywhere in mathematics. A mathematical system consists of a set S and certain *constants* associated with S. These constants may be of various logical types: elements of S, subsets of S, families of subsets (as a topology), functions, operations, *etc.* Any one to one transformation of S onto itself gives rise in obvious ways to induced transformations of all these logical types, so maps each constant into an object of the same type. Those which take each given constant into itself are called *automorphisms* (or *symmetries*) of the system, and form a group G. The *structure* of the system is then defined as the aggregate of all objects, of whatever logical type, which are left fixed by G. There is a one to one correspondence between groups and structures, the richer the structure the smaller the group.

A transformation group G is *transitive* if given a and b in S there exists f in G such that $f(a)=b$. G is *simply transitive* if this f is unique for each a and b. In a sense transitivity keeps G from being very small, while *simple* transitivity keeps G from being very large. Among groups which are transitive but not simply transitive are doubly transitive groups. A transformation group G is *doubly transitive* if given $a_1\neq a_2$ and $b_1\neq b_2$ in S, there exists f in G such that

* A somewhat different proof of the theorem of this paper was submitted by the second author to Wesleyan University Honors College as an undergraduate Honors Thesis, 1959.

$f(a_1)=b_1$ and $f(a_2)=b_2$. We shall call such a group *just doubly transitive* if the f is unique for each a_1, a_2, b_1, b_2. (The restriction $a_1 \neq a_2$, $b_1 \neq b_2$ is unavoidable, because f must be one to one.) Similarly *n-transitivity* and *just n-transitivity* can be defined for each nonnegative integer n. (All groups are 0-transitive, while only the trivial group is just 0-transitive.) It is unusual, however, for an automorphism group to be just n-transitive for some n. Examples: The automorphism group of the real line is just 0-transitive. The group of translations in the real line (or, for that matter, in any abstract group) is simply transitive, *i.e.*, just 1-transitive. The group of affine functions $f(x)=ax+b$ $(a \neq 0)$ of the real line (or for that matter any field) is just doubly transitive. The group of projectivities of a projective line is just triply transitive. The group of homeomorphisms of a plane (but not of a line) is n-transitive for every n.

Our axiomatic description of the line can be stated thus: *the line is a set S having enough structure so that its automorphism group G is just doubly transitive, and having as part of that structure a complete (in the Dedekind sense) betweenness order.* We shall prove (see more explicitly the theorem in §2) that all such systems are isomorphic.

Axioms for the *ordered line* are well known [4]. Stated briefly they are that the order be complete, unbounded, and dense, and that S have a countable dense subset. Our line is this ordered line so far as its order alone is concerned, but as remarked earlier, it has more structure not definable in terms of order. It is interesting that the existence of such structure, expressed by G, is enough to make redundant most of the above order axioms.

2. Formulation of the problem. We begin by establishing notation and stating the theorem to be proved in detail. We are given a set S, with elements $a, b, \cdots$, ordered by a betweenness relation. We can dispense with restating the axioms for betweenness (see [3]) because betweenness gives rise to order, $<$, of the ordinary familiar kind as soon as an arbitrary choice of direction in S is made, for example by stipulating $a_0<a_1$ for two arbitrarily selected points a_0 and a_1. Only two such orders arise (the other by stipulating instead $a_1<a_0$); we adopt one of them arbitrarily as a notational device and state the order axioms in terms of $<$, as follows:

2.1 (Transitive law) *If $a<b$ and $b<c$ then $a<c$.*

2.2 (Law of Trichotomy) *For any a and b one and only one of the following is true: $a<b$, $a=b$, $b<a$.*

2.3 (Completeness in the sense of Dedekind) *Any nonempty subset A of S having an upper bound has a least upper bound, denoted by* sup A.

It is well known that 2.3 implies its dual, so we have greatest lower bounds, denoted inf A, as well.

The assumption of just double homogeneity of S is embodied by a given family G of functions of S onto S governed by the following axioms:

2.4 *G is a transformation group.*

2.5 *The functions in G preserve betweenness, that is, if* $a<b<c$, *then either* $f(a)<f(b)<f(c)$ or $f(c)<f(b)<f(a)$.

2.6 *Given* $a\neq a'$ *and* $b\neq b'$, *there exists in G a unique function f such that* $f(a)=b$ *and* $f(a')=b'$.

Finally, to give Axiom 2.6 force we need a nontriviality axiom:

2.7 *S has at least three elements.*

The most obvious example, hereinafter referred to as the *prototype*, of a system satisfying these axioms is the real line, namely the set of all real numbers with the usual order, and with the family of all affine functions $ax+b$ $(a\neq 0)$ for G. Our conclusion will be that any system satisfying the axioms is isomorphic to the prototype. More explicitly we shall prove:

THEOREM. *Given* $a_0<a_1$ *in S, there exists a unique one to one correspondence* ϕ *of the real line onto S such that*

1) $\phi(0)=a_0$, $\phi(1)=a_1$ *and*

2) *the correspondence* $f\to\phi f\phi^{-1}$ *of real functions of a real variable to functions of S to S maps the family of affine functions onto G.*

Moreover this correspondence ϕ *is order preserving.*

The group operation in Axiom 2.4 is of course composition, and the functions of which G is made up must have inverses in G, so must be one to one. The correspondence used in 2) of the theorem is not capricious, but standard; it is designed so that if $\phi(x)=a$ and $\phi(y)=b$, and if f takes x into y, then the corresponding function takes a into b. This correspondence, if it satisfies 2), is automatically an isomorphism of the two groups.

Some easy consequences of the axioms can be drawn immediately. By the uniqueness in 2.6, any function in G which leaves two points fixed must be the identity, because the identity leaves those points (in fact all points) fixed.

2.8 *The order in S is unbounded, that is, no c in S is maximum or minimum.*

Proof. Given c, there are two other points a and b by 2.7. If $a<c<b$ there is nothing to prove, so suppose for example that $a<b<c$. Let f be the function in G such that $f(a)=b$ and $f(b)=c$. Then by 2.5, $f(a)<f(b)<f(c)$, *i.e.*, $c<f(c)$. Thus c is not minimum or maximum. The remaining case $c<a<b$ is similar.

2.9 *The order in S is dense.*

Proof. Similar to 2.8.

2.10 *Every function in G is either order preserving or order reversing.*

Proof. Clear from 2.5.

The uniqueness of the correspondence ϕ asserted in the theorem can be

proved quite easily, and we might as well dispose of this now. Suppose that ϕ and ψ are one to one correspondences of the real line onto S, both satisfying 1) and 2). Then the composite function $g=\psi^{-1}\phi$ has the following properties:

(a) g is one-to-one and maps the real line onto itself.
(b) $g(0)=0$, $g(1)=1$.
(c) If f is any affine function, then $gfg^{-1}=\psi^{-1}(\phi f\phi^{-1})\psi$ is an affine function.

We want to prove that g is the identity: $g(x)=x$ for all x. We begin by applying (c) to the particular affine function $f(x)=ax$, $a\neq 0$, obtaining $g(ag^{-1}(x))=bx+c$, identically in x for some b and c. Setting successively $x=0$ and $x=1$ and using (b) we find $c=0$ and $b=g(a)$. Then putting u for $g^{-1}(x)$, we get the functional equation $g(au)=g(a)\cdot g(u)$, which holds for all a and u (trivially for $a=0$). In particular this gives $g(-1)^2=g(1)=1$, whence using (a) and (b) $g(-1)=-1$. Now apply (c) again, this time to the affine function $f(x)=x+1$. A similar evaluation of the constants leads to the functional equation $g(u+1)=(g(2)-1)g(u)+1$. In particular, $u=-1$ leads to the evaluation $g(2)=2$, whence $g(u+1)=g(u)+1$. Now for $c\neq 0$, $g(u+c)=g((uc^{-1}+1)c)=g(uc^{-1}+1)g(c)=[g(uc^{-1})+1]g(c)=g(uc^{-1})g(c)+g(c)=g(u)+g(c)$. Since this functional equation is also trivially true for $c=0$, g turns out to be a homomorphism of the real number field, and not identically 0. There is only one such homomorphism, and that is the identity function [1, p. 47].

3. Inversions and fixed points. Denote the identity function on S by i: $i(a)=a$ for $a\in S$. A function in G other than the identity which is its own inverse, *i.e.*, such that $f^2=i$, is called an *inversion*. Given any pair of distinct points, the function in G which interchanges them is an inversion, since its square leaves both points fixed. Clearly inversions are order reversing.

3.1 *If $f\in G$ and $a<f(a)$, $f(b)<b$ then f has a fixed point between a and b.*

Proof. Case I: f is order preserving. We can suppose $a<b$, for the other case reduces to this by interchanging a and b and replacing f by f^{-1}. Since f is order preserving we have $a<f(a)<f(b)<b$. Let A be the set of those x such that $a\leqq x<b$ and $x<f(x)$. Then $a\in A$ and A is bounded, so $c=\sup A$ exists. Suppose $c<f(c)$. Then $c<b$ and there exists x such that $c<x<b$ and $x<f(c)$. Since f is order preserving $f(c)<f(x)$, so that $x\in A$, contradicting $c<x$. On the other hand suppose $f(c)<c$. Then there exists x in A such that $f(c)<x\leqq c$. Since f is order preserving, $f(x)\leqq f(c)<x$, contradicting $x\in A$. Thus $f(c)=c$ and c is a fixed point.

Case II: f is order reversing. In this case certainly $a<b$, for $b<a$ would give $f(b)<f(a)$, making f order preserving. Let A be the set of all x such that $x<f(x)$. Then $a\in A$ and b is an upper bound for A. Indeed $b<x$ implies $f(x)<f(b)<b<x$, whence $x\notin A$. Thus $c=\sup A$ exists. Note that since f and hence also f^{-1} are order reversing, for any x other than a fixed point (we can assume there are none) either x or $f^{-1}(x)$ belongs to A. Indeed, $f(x)<x$ implies $f^{-1}(x)<f^{-1}(f(x))$

$=x=f(f^{-1}(x))$. Now suppose $c<f(c)$ and find x such that $c<x<f(c)$. Then applying f^{-1}, $c<f^{-1}(x)$. But either x or $f^{-1}(x)$ belongs to A, and $c=\sup A$, a contradiction. On the other hand suppose $f(c)<c$ and find $x\in A$ so that $f(c)<x<f(x)$. Applying f^{-1} get $x<f^{-1}(x)<c$. Now find y in A so that $f^{-1}(x)<y$. Then $y<f(y)<x<f^{-1}(x)<y$, a contradiction. Thus finally $c=f(c)$, and c is a fixed point.

3.2. COROLLARY. *Every inversion has a fixed point* (*unique by* 2.6).

3.3. COROLLARY. *If* $f\in G$, $a\leqq f(a)$, $b<f(b)$ *and* $a<c<b$, *and* $c\leqq f(c)$.

Proof. Otherwise f would have to have two fixed points.

We shall need the following sharpening of 3.3.

3.4 *If* $f\in G$, $a\leqq f(a)$, $b<f(b)$ (*or* $a<f(a)$, $b\leqq f(b)$) *and if* $a<c<b$, *then* $c<f(c)$.

Proof. Suppose first that $f(b)<f(a)$. Then f is order reversing, and $c<b$ leads to $f(b)<f(c)$, whence $c<f(c)$. Assume then that f is order preserving. By 3.3 we need only rule out the eventuality $f(c)=c$, which is certainly impossible if $a=f(a)$. Assume, then, that $a<f(a)$ and $f(c)=c$. Let g be the function in G such that $g(f(a))=a<f(a)$ and $g(f(b))=f(b)$. Then $c<g(c)$ would require g to have another fixed point between $f(a)$ and c (where $f(b)$ isn't), and $c=g(c)$ likewise would give g two fixed points. Hence $g(c)<c$. Now the function gf leaves a fixed and also (since $gf(b)=f(b)>b$ and $gf(c)=g(c)<c$) has another fixed point between b and c, an impossibility.

3.5 *Each point* c *of* S *is the fixed point of a unique inversion in* G.

Proof. Let $a<c$. For each $x\neq c$ there is a unique function f_x in G such that $f_x(c)=c$ and $f_x(a)=x$. In fact $x\to f_x$ is a one to one correspondence of $S-\{c\}$ to the subgroup of G leaving c fixed. Evidently any solution of the quadratic equation $f_x(x)=a$, other than the trivial one $x=a$, corresponds to an inversion with c as fixed point, and conversely.

We shall consider only f_x for $x>c$; each such f_x is order reversing. Let $x'=f_x^{-1}(a)$. Since $f_{x'}$ and f_x^{-1} agree at a and c, they are equal. It follows that if $a<f_x(x)$, then $x=f_{x'}f_x(x)<f_{x'}(a)=x'$ and also conversely. Thus $a<f_x(x)$, $x<x'$ and $f_{x'}(x')<a$ are equivalent. Evidently $x=x'$ gives $f_x(x)=a$ and solves the problem.

Suppose $c<x<y$ and $c<u$. We shall prove that $f_y(u)<f_x(u)$. Assume on the contrary that $f_x(u)\leqq f_y(u)$ and consider the function $g=f_yf_{x'}$. Evidently $g(x)=y>x$ and $g(f_x(u))=f_y(u)\geqq f_x(u)$. Since $f_x(u)<c<u$ and $g(c)=c$, we have a contradiction of 3.4. We conclude that $f_y(u)<f_x(u)$.

Two applications of this result: First, taking $u=x'$, we get $f_y(x')<a$, whence $y'<x'$. Thus $x<y$ implies $y'<x'$. Secondly, take $u=x$ to get $f_y(x)<f_x(x)$. Since $f_y(y)<f_y(x)$ we have that $x<y$ implies $f_y(y)<f_x(x)$. This, incidentally, insures the uniqueness of the solution.

Now let A be the set of all $x>c$ such that $f_x(x)>a$. We have shown that for

any $x>c$ either x or x', not both, belongs to A, and therefore A is not empty. Also if $x\in A$, then x' is an upper bound for A, by the second result of the last paragraph. Hence $b=\sup A$ exists. If b and b' are not equal, then there exists x between them, and by the first result of the last paragraph, x' also lies between them. Then one of x and x' belongs to A, while the other is an upper bound for A, which is impossible because $b=\sup A$. Hence $b=b'$ is a solution.

3.6 *Let $f\in G$, let $f(a_1)=a_2$ and $f(b_1)=b_2$. Let c_i be the fixed point of the inversion in G which interchanges a_i and b_i $(i=1, 2)$. Then $f(c_1)=c_2$.*

Proof. Call the two inversions in question f_1 and f_2 and let $g=f_2ff_1$. Then $g(a_1)=a_2=f(a_1)$ and $g(b_1)=b_2=f(b_1)$ so that $g=f$, *i.e.*, $f_2f=ff_1$. Then $f_2(f(c_1))=f(f_1(c_1))=f(c_1)$, so that $f(c_1)$ is a fixed point of f_2. Since f_2 has only one fixed point, $f(c_1)=c_2$.

3.7 *Let $f\in G$, let $f(a_1)=a_2$, $f(c_1)=c_2$. Let $b_i=f_i(a_i)$, where f_i is the inversion in G with c_i for fixed point $(i=1, 2)$. Then $f(b_1)=b_2$.*

Proof. Let g be the function in G such that $g(a_1)=a_2$ and $g(b_1)=b_2$. Then by 3.6 $g(c_1)=c_2$. Since g agrees with f at a_1 and c_1, $g=f$, so $f(b_1)=g(b_1)=b_2$.

4. Ladders and ramps. Our information about inversions can be summarized in an algebraic way by defining two operations in S, inverse to each other, as follows:

Interpolation: *define $(a*b)$ to be the fixed point of the inversion in G which interchanges a and b.*

Extrapolation: *define $(ac*)$ to be the image of a under the inversion in G which has c for its fixed point.*

Then we have evidently the following algebraic rules:

4.1 $(a*a)=(aa*)=a$.

4.2 $(a*b)=(b*a)$; $(ab*)=c$ *if and only if* $(cb*)=a$.

4.3 $(a*b)=c$ *if and only if* $(ac*)=b$.

Using this terminology, the content of 3.6 and 3.7 is that functions in G preserve interpolation and extrapolation. Applying this in particular to an inversion we have:

4.4 *If* $(a*a')=(b*b')=c$, *then also*
(1) $(a*b)*(a'*b')=c$ *and*
(2) $(ab*)*(a'b'*)=c$.

We shall use this property of functions in G to propagate information about them from a set consisting of only two points to a much larger set, in fact a set large enough to be decisive, as will come out in the next section.

We shall call a subset L of S a *ladder* when L is closed under extrapolation: $a\in L$ and $b\in L$ imply $(ab*)\in L$. Given any two points a_0, a_1 (more generally any subset) there exist ladders containing a_0 and a_1, for example S; the intersection of all such ladders is a ladder, called the ladder *generated* by a_0, a_1.

In our prototype, the real line, the ladder generated by two numbers is the both-ways-infinite arithmetic progression with these two numbers as adjacent terms, for example the integers. In the integers regarded as a ladder the operation is $(mn*)=2n-m$.

4.5 *Given $a_0 \neq a_1$ in S, there exists a unique isomorphism $n \to a_n$ of the real ladder of integers onto the ladder in S generated by a_0, a_1 such that $0 \to a_0$ and $1 \to a_1$.*

Proof. For $n \to a_n$ to be an isomorphism means that $a_{2n-m}=(a_m a_n *)$ must be satisfied for all integers m, n. Writing p for $n-m$ and using 4.3 this can be put in the more symmetric form $(a_{n-p} * a_{n+p})=a_n$. In particular this requirement dictates the following recursion formulae

$$a_{n+1} = (a_{n-1}a_n *) \qquad n = 1, 2, \cdots$$
$$a_{n-1} = (a_{n+1}a_n *) \qquad n = 0, -1, \cdots$$

which together with the given values of a_0 and a_1 determine the correspondence uniquely.

To show that the correspondence thus recursively defined is an isomorphism, it is sufficient to verify $(a_{n-p} * a_{n+p})=a_n$ for all n and p. Do this by induction on p. For $p=0$ it is trivial, and for $p=1$ it is the recursive condition. Suppose $(a_{n-p} * a_{n+p})=a_n$ for $p=0, 1, \cdots, k$, in particular for $p=k-1$ and $p=k$. Since $a_{n-k-1}=(a_{n-k+1}a_{n-k} *)$ and $a_{n+k+1}=(a_{n+k-1}a_{n+k} *)$ by definition, 4.4 (2) gives $(a_{n-k-1} * a_{n+k+1})=a_n$, completing the induction. Finally, the correspondence being an isomorphism, its range must be a ladder, clearly the ladder generated by a_0, a_1.

Obviously 4.5 could have been stated more generally for any arithmetic progression in place of the integers, at some cost in notation. The same applies to the next lemma, which is in preparation for what follows:

4.6 *Given an isomorphism $2m \to a_{2m}$ of the real ladder of the even integers onto a ladder in S, it can be uniquely extended to an isomorphism of the ladder of all integers.*

Proof. As in 4.5, the condition which must be satisfied by the extended correspondence is $(a_{n-p} * a_{n+p})=a_n$ for all n, p. In particular this uniquely dictates the definition $a_{2m+1}=(a_{2m} * a_{2m+2})$ for the odd integers $2m+1$, and the uniqueness is proved. To see that this does give an isomorphism we can use the proof of 4.5 once we verify $(a_{n-1} * a_{n+1})=a_n$. For n odd this was the definition. For $n=2m$ even, it reads $(a_{2m-1} * a_{2m+1})=a_{2m}$. This is an application of 4.4 (1), since $(a_{2m} * a_{2m})=a_{2m}$ trivially and $(a_{2m-2} * a_{2m+2})=a_{2m}$ is given by hypothesis.

When the rungs of a ladder are very close together, it is a ramp. Explicitly we shall call a subset R of S a *ramp* when R is closed under extrapolation and interpolation: $a \in R$ and $b \in R$ imply $(ab*) \in R$ and $(a * b) \in R$. Given any two points a_0, a_1 there exist ramps containing them, for example S; the intersection of all such ramps is a ramp, called the ramp *generated* by a_0, a_1. This ramp is also a ladder, so contains the ladder generated by a_0, a_1.

In our prototype, the real line, the ramp generated by 0, 1 consists of all the dyadic rationals $m2^{-n}$. In the dyadic rationals regarded as a ramp, the operations are of course $(rs*)=2s-r$ and $(r*s)=\frac{1}{2}(r+s)$.

4.7 *Given $a_0 \neq a_1$ in S, there exists a unique isomorphism $r \to a_r$ of the real ramp of the dyadic rationals onto the ramp in S generated by a_0, a_1 such that $0 \to a_0$ and $1 \to a_1$.*

Proof. The isomorphism is uniquely determined for integers r by 4.5. It is then uniquely determined for dyadic rationals with higher and higher denominators by repeated use of 4.6. The induction hinted at here proves the uniqueness. This also defines recursively a correspondence $r \to a_r$ whose domain in the limit includes all dyadic rationals. We have seen in the proof of 4.5 that in proving this correspondence is an isomorphism, both operations are disposed of at once by verifying the one formula $(a_{r-s} * a_{r+s}) = a_r$. But any instance of this formula involves only dyadic rationals with a single denominator 2^{-n}, the common denominator of r and s. From 4.6 applied n times, the correspondence is an isomorphism of the ladder of multiples of 2^{-n}, so this instance of the formula is true. Finally, the correspondence being an isomorphism, its range is a ramp, clearly the ramp generated by a_0, a_1.

5. Archimedean properties. As a result of Section 4, given two points a_0 and a_1 we know completely the structure of the ramp R they generate, and according to 3.6 and 3.7 any function in G is determined on R by its values at a_0 and a_1. To complete the picture we need to know the relationship of R to the rest of S, and this is a topological relationship (based on order) rather than an algebraic one.

The obvious properties of extrapolation and interpolation in relation to order are:

5.1 *If $a<b$, then $a<(a*b)<b$ and $b<(ab*)$.*

5.2 (a) *If $a \leqq a'$ and $b<b'$ then $(a*b)<(a'*b')$.*
(b) *If $b<b'$ then $(ab*)<(ab'*)$.*

Proof. To prove (a), let f be the function in G such that $f(a)=a'$ and $f(b)=b'$. Then by 3.6 $(a'*b')=f(a*b)$, and using 5.1 and 3.4, $(a*b)<(a'*b')$. Now (b) follows easily from (a).

5.3 *If $a_0<a_1$, then the ladder and ramp isomorphisms of* 4.5 *and* 4.7 *are order-preserving.*

Proof. An inequality $a_{n-1}<a_n$ and the recursion relation $a_{n+1}=(a_{n-1}a_n*)$ together with 5.1 imply $a_n<a_{n+1}$. Hence by induction $a_n<a_{n+1}$ for $n=0, 1, \cdots$, and similarly for the negative integers. Thus the isomorphism of 4.5 is order preserving. It is clear that each extension of the isomorphism by an application of 4.6 is still order-preserving. Therefore the ramp isomorphism of 4.7 is also.

5.4 *Every ladder containing at least two points is unbounded above and below.*

Proof. Let L be a ladder containing two points, say a and b, and suppose that $c=\sup L$ exists. Now $c>a$ because either $a<b$ or else $a<(ba*)\in L$. Hence $a<(a*c)<c$. Thus there exists $x\in L$ such that $(a*c)<x$. But then by 5.2 (b) $c=(a(a*c)*)<(ax*)$, while $(ax*)\in L$, a contradiction. Existence of $\inf L$ is ruled out similarly.

5.5 *Every ramp R containing at least two points is dense; that is, given $b_0<b_1$ in S, there exists a in R such that $b_0<a<b_1$.*

Proof. Let L be the ladder generated by b_0, b_1 and let $n\to b_n$ be the isomorphism of the integers onto L such that $0\to b_0$ and $1\to b_1$. This correspondence is order preserving by 5.3. Now R and L being both unbounded (5.4) there are elements of R greater than b_0, and then elements of L greater than these. Hence there is a largest integer n such that no r in R satisfies $b_0<r<b_n$. Similarly find the smallest (negative) integer m such that no r in R satisfies $b_m<r<b_0$. We want to prove $n=0$. If $n>0$ then we have $b_{n-1}<r\leqq b_n<b_n\leqq s<b_{n+1}$ for some r and s in R, and no point of R lies between b_n and b_n. But on the contrary we shall show that $(r*s)$ is such a point. Indeed, $(r*s)\in R$ because R is a ramp. By two applications of 5.2 (a) $(r*s)<(b_m*b_{n+1})\leqq(b_{n-1}*b_{n+1})=b_n$, and similarly $b_m<(r*s)$. This contradiction proves that R is dense.

6. Completion of the theorem. We can now finish the proof of the theorem of Section 2. We start with $a_0<a_1$ in S and define $\phi(x)$ for dyadic rationals x to be the ramp isomorphism determined by 4.7 of the dyadic rationals onto the ramp generated by a_0, a_1. According to 5.3 and 5.5 ϕ is an order-preserving function mapping a dense subset of the real line onto a dense subset of S. Since S is complete, it is well known that ϕ has a unique one-to-one order-preserving extension (still denoted ϕ) mapping all of the real line onto all of S. (For x irrational $\phi(x)$ is defined as the least upper bound of $\phi(r)$ for r dyadic rational, $r<x$. See [4] p. 49.) We must prove that this ϕ satisfies 2) of the theorem.

Let f be an affine function, and let $b_0=\phi f(0)$, $b_1=\phi f(1)$. Let g be the function in G such that $g(a_0)=b_0$ and $g(a_1)=b_1$. Then g and $\phi f\phi^{-1}$ are functions of S onto S, and they agree at a_0 and a_1. Moreover both these functions preserve extrapolation and interpolation on the ramp R generated by a_0 and a_1, g because it belongs to G, and $\phi f\phi^{-1}$ because ϕ is an isomorphism of this ramp and f is affine. Hence g and $\phi f\phi^{-1}$ agree on R. Finally g and $\phi f\phi^{-1}$ both preserve betweenness in S, g because it belongs to G (Axiom 2.5), and $\phi f\phi^{-1}$ because ϕ is order-preserving and f is affine. Since R is dense, g and $\phi f\phi^{-1}$ agree everywhere, that is $\phi f\phi^{-1}\in G$. The same argument with the roles of S and the real line interchanged shows that every function in G corresponds to an affine function, and the proof is complete.

7. Final remarks. The theorem of Section 2 has a counterpart for the projective line. The prototype is in this case the *real projective line*, whose points are the real numbers and ∞, and whose automorphisms are the real projective functions

$$f(x) = \frac{ax + b}{cx + d} \qquad ad - bc \neq 0.$$

The order involved is a circular order [2] defined by a quaternary relation "*a* and *b* *separate* *c* and *d*".

THEOREM. *Let S be a set with a complete circular order, G a just triply transitive transformation group on S preserving circular order. Then given any three distinct points* a_0, a_1, a_∞ *of S there is a unique one-to-one correspondence* ϕ *of the projective real line onto S such that*

1) $\phi(0) = a_0$, $\phi(1) = a_1$, $\phi(\infty) = a_\infty$ *and*

2) $f \to \phi f \phi^{-1}$ *maps the family of real projective functions onto G.*

Moreover ϕ *is circular order-preserving.*

We shall sketch a proof based on the affine theorem. The group G_0 which leaves a_∞ fixed acts just doubly transitively on the set $S_0 = S - \{a_\infty\}$. The circular order on S gives a betweenness order on S_0 which is preserved by G_0. Hence applying the affine theorem to S_0 with G_0 gives a unique betweenness preserving correspondence ϕ of the real line onto S_0 such that $\phi(0) = a_0$, $\phi(1) = a_1$ and the affine functions are mapped onto G_0. Extend ϕ by defining $\phi(\infty) = a_\infty$.

It remains to prove that the rest of the real projective functions correspond to the rest of G. This follows easily as soon as we show that the reciprocal function $1/x$ goes into a function in G. To prove this, note that there is a function g in G such that $g(a_0) = a_\infty$, $g(a_\infty) = a_0$ and $g(a_1) = a_1$. Let $f = \phi^{-1} g \phi$. Then f is a real function satisfying $f(0) = \infty$, $f(\infty) = 0$ and $f(1) = 1$, and it suffices to prove that $f(x) = 1/x$ identically. Now for any $a \neq 0$ the function $h(x) = f(ax)/f(a)$ is the composition of f with two affine functions, whence $\phi h \phi^{-1}$ belongs to G. But by computation $\phi h \phi^{-1}$ has the same effect on a_0, a_1 and a_∞ as g has, whence $\phi h \phi^{-1} = g$, $h = f$. This gives the functional equation $f(ax) = f(a) f(x)$. The only circular order preserving solutions of this equation are $f(x) = x^n$ for $n \neq 0$. Now since g^2 has a_0, a_1 and a_∞ as fixed points, g, and therefore also f, are involutions. We have $(x^n)^n = x$, and $n = \pm 1$. Since f is clearly not the identity $n = -1$ and $f(x) = 1/x$.

These characterizations appear to be special to one dimension. The obstacle in more dimensions is not only the problem of finding a substitute for order, but worse, the fact that the group of nonsingular affine transformations in n dimensions is not quite $(n+1)$-transitive: $a_0, a_1, \cdots, a_n$ can be mapped onto arbitrary $b_0, b_1, \cdots, b_n$ only when the a's and the b's satisfy the same coincidences of collinearity, for example when both sets are in general position. Thus the group cannot be adequately described without already having a structure of straight lines. But the straight line structure of the space is enough to determine the group, so mention of the group would be superfluous. Indeed we can say, rather, only in one dimension is a characterization like the one given needed.

In fact, in the light of the last remark one would say that the group G re-

garded as structure of the affine line is inherited from the more-dimensional geometry in which the line can be embedded, its just double transitivity being a consequence of the incidence axioms of that geometry. The presence of order should then be regarded as a further axiomatic restriction on the geometry (or alternatively on the group) which selects the real affine geometry from among all abstract affine geometries. This raises the interesting question: what is the relation between just doubly transitive groups and affine geometries in general? On this question see R. D. Carmichael, *Groups of Finite Order*, Chapters XI–XIII.

References

1. E. Artin, Geometric Algebra, New York, 1957.
2. H. M. S. Coxeter, The Real Projective Plane, Cambridge, 1955.
3. D. Hilbert, Foundations of Geometry (Translated by E. J. Townsend), LaSalle, Ill. 1950.
4. E. V. Huntington, The Continuum, Cambridge, Mass. 1921.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

L. E. BUSH, Kent State University

The following results of the twentieth William Lowell Putnam Mathematical Competition held on November 21, 1959, have been determined in accordance with the constitution of the competition. This competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, five hundred dollars, is awarded to the Department of Mathematics of the Polytechnic Institute of Brooklyn, Brooklyn, New York. The members of the team were Martin Isaacs, Donald Passman, and Gerald Stoller; to each of these a prize of fifty dollars is awarded.

The second prize, four hundred dollars, is awarded to the Department of Mathematics of the California Institute of Technology, Pasadena, California. The members of the team were Donald W. Anderson, Alfred W. Hales, and Harold M. Stark; to each of these a prize of forty dollars is awarded.

The third prize, three hundred dollars, is awarded to the Department of Mathematics of the University of Toronto, Toronto, Ontario. The members of the team were Edward J. Barbeau, Andrew Korsak, and Joseph Lipman; to each of these a prize of thirty dollars is awarded.

The fourth prize, two hundred dollars, is awarded to the Department of Mathematics of Harvard University, Cambridge, Massachusetts. The members of the team were Stephen L. Adler, Stephen Lichtenbaum, and Robert Wells; to each of these a prize of twenty dollars is awarded.

The fifth prize, one hundred dollars, is awarded to the Department of Mathematics of Case Institute of Technology, Cleveland, Ohio. The members

of the team were Daniel J. Gans, Donald E. Knuth, and David F. Monck; to each of these a prize of ten dollars is awarded.

The eight persons ranking highest in the examination (the ranks five to eight being tied), named in alphabetical order, are Stephen L. Adler, Harvard University; Donald Gorman, Harvard University; Alfred W. Hales, California Institute of Technology; Martin Isaacs, Polytechnic Institute of Brooklyn; Samuel Klein, College of the City of New York; Stephen Lichtenbaum, Harvard University; Donald Passman, Polytechnic Institute of Brooklyn; and Daniel G. Quillen, Harvard University. Each of these will receive a prize of seventy-five dollars, and the three-thousand-dollar William Lowell Putnam Prize Scholarship at Harvard will be awarded to one of this number.

The two succeeding persons ranking highest in the examination, named in alphabetical order, are Alan Landman, Harvard University; and Harold M. Stark, California Institute of Technology. Each of these will receive a prize of thirty-five dollars.

The following teams, named in alphabetical order, won honorable mention: College of the City of New York, New York, New York, the members of the team being Stephen Berger, Samuel Klein, and Eugene Luks; Dartmouth College, Hanover, New Hampshire, the members of the team being William A. Veech, Albert R. Whitcomb, and Seth I. Zimmerman; Massachusetts Institute of Technology, Cambridge, Massachusetts, the members of the team being Richard L. Faber, Steven A. Orszag, and Stephen Scheinberg; Oberlin College, Oberlin, Ohio, the members of the team being Stephen A. Andrea, Christopher Billings, and Peter J. Kahn; Yale University, New Haven, Connecticut, the members of the team being Richard William Beals, Peter Diamond, and Theodore William Gamelin; and Yeshiva University, New York, New York, the members of the team being Stanley Boylan, Jonathan Ginsberg, and Benjamin Weiss.

Eighteen individuals were given honorable mention. The names, alphabetically arranged, are: Donald W. Anderson, California Institute of Technology; Edward J. Barbeau, University of Toronto; Richard William Beals, Yale University; Stefan Burr, University of California, at Berkeley; B. P. Carter, Stanford University; William Easton, Cornell University; Jon Folkman, University of California, at Berkeley; Donald E. Knuth, Case Institute of Technology; Andrew Korsak, University of Toronto; Joseph Lipman, University of Toronto; John Lovin, Reed College; Peter T. Saunders, University of Toronto; Stanley A. Sawyer, California Institute of Technology; Jack Silver, Montana State University; Gerald Stoller, Polytechnic Institute of Brooklyn; Floris Tsang, University of California, at Berkeley; James Wirth, University of Notre Dame; and Seth I. Zimmerman, Dartmouth College.

A total of eight hundred and twenty-nine contestants from one hundred and forty-one colleges and universities (one hundred and twelve of these having teams) entered the Competition. Six hundred and thirty-three contestants from one hundred and thirty-five colleges and universities (ninety-five having teams)

participated in the examination on November 21, 1959.

The following is a list, in alphabetical order, of all colleges and universities which entered teams in the competition:

Agricultural and Mechanical College of Texas, Alabama College, American University, Arizona State College (Flagstaff), Arizona State University (Tempe), Austin College, Bethel College, Brooklyn College, California Institute of Technology, Carleton College, Carnegie Institute of Technology, Case Institute of Technology, Central Michigan University, College of Saint Catherine, College of Saint Thomas, College of the City of New York, College of the Holy Cross, Columbia College, Cornell University, Dartmouth College, Eastern Illinois University, Eastern New Mexico University, Fairleigh-Dickinson University, Fordham College, George Pepperdine College, Georgia Institute of Technology, Glenville State College, Grambling College, Harvard University, Iowa State University, Iowa Wesleyan College, Ithaca College, Kenyon College, Kent State University, Lafayette College, Lebanon Valley College, Long Beach State College, Mankato State College, Massachusetts Institute of Technology, McGill University, McMaster University, Memphis State University, Michigan State University, Mississippi State University, Northeastern University, Oberlin College, Ohio State University, Oregon State College, Pennsylvania State University, Polytechnic Institute of Brooklyn, Pomona College, Purdue University, Queen's University (Ontario), Reed College, Rice Institute, Rutgers University, Sacramento State College, Saint Francis Xavier University, Saint Joseph's College (West Hartford, Connecticut), Saint Martin's College, San Jose State College, Seattle University, Siena College, Shaw University, South Dakota School of Mines, Southern University, Southwestern at Memphis, Southwestern Louisiana Institute, Stanford University, State University of Iowa, Swarthmore College, Texas Technological Institute, Valparaiso University, Vanderbilt University, United States Naval Academy, University of Alberta, University of Arizona, University of British Columbia, University of Buffalo, University of California (Berkeley), University of California (Davis), University of California (Los Angeles), University of Colorado, University of Dayton, University of Idaho, University of Illinois, University of Kansas, University of Kansas City, University of Manitoba, University of Michigan, University of Minnesota, University of Nebraska, University of Notre Dame, University of Ottawa, University of Pennsylvania, University of Rochester, University of Santa Clara, University of Tampa, University of Toronto, University of Washington, University of Western Ontario, University of Wichita, University of Wisconsin, Ursinus College, Washington State University (Pullman), Washington University (Saint Louis), Wayne State University, Wesleyan University, William Jewell College, Wofford College, Yale University, and Yeshiva University.

The following colleges and universities, in alphabetical order, entered individual contestants only:

Anderson College, Assumption University (Ontario), Bennington College, Brandeis University, Caldwell College, Colorado College, Colorado State University, Drew University, Eastern Baptist College, Fresno State College, Grinnell College, Lake Forest College, Lindenwood College, Manhattan College, Montana State University, Newark College of Engineering, New Mexico State University, North Carolina State College, Parsons College, Saint Joseph's College (Philadelphia), Saint Mary's College, Saint Olaf College, Stevens Institute of Technology, University of Chicago, University of Dallas, University of Maine, University of the South, University of Vermont, and Wake Forest College.

The individual rankings of contestants (except for the relative ranks of the first eight) may be obtained by any department of mathematics for the purpose of selecting graduate students.

The problems given to those participating in the competition, together with a write-up of the solutions, will appear in a later issue of the MONTHLY.

MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

Material for the department should be sent to Roy Dubisch, Department of Mathematics, University of California, Berkeley 4, California.

REMARKS ON UNIFORM CONTINUITY IN METRIC SPACES

NORMAN LEVINE, University of Pittsburgh

Every continuous transformation from one metric space into another is in a certain sense uniformly continuous. The sense in which this is so is implied in the following

THEOREM 1. *Let $f: S_1 \to S_2$ be a continuous transformation where S_1 and S_2 are metric spaces with metrics d_1 and d_2 respectively. Then there exists a metric d_1^* on S_1 which is equivalent to d_1 such that $f: S_1 \to S_2$ is uniformly continuous relative to d_1^* and d_2.*

Proof. Let $d_1^*(a, b) = d_1(a, b) + d_2(f(a), f(b))$. We will show that (1) $d_1^*(a, b)$ is single valued and nonnegative; (2) $d_1^*(a, b) = d_1^*(b, a)$; (3) $d_1^*(a, b) = 0$ if and only if $a = b$; (4) $d_1^*(a, b) + d_1^*(b, c) \geqq d_1^*(a, c)$; (5) $\lim d_1^*(y_n, y) = 0$ if and only if $\lim d_1(y_n, y) = 0$ for every sequence $\{y_n\}$ and every point y in S_1; (6) $f: S_1 \to S_2$ is uniformly continuous relative to d_1^* and d_2. (1) and (2) are clear. To show (3) let $d_1^*(a, b) = 0$. Then $d_1(a, b)$ is 0 and hence $a = b$. Conversely let $a = b$. Then $d_1(a, b)$ and $d_2(f(a), f(b))$ are both 0 and hence $d_1^*(a, b) = 0$. Now $d_1^*(a, b) + d_1^*(b, c) = d_1(a,b) + d_2(f(a),f(b)) + d_1(b,c) + d_2(f(b),f(c)) \geqq d_1(a,c) + d_2(f(a),f(c)) = d_1^*(a, c)$. This proves (4). We now prove (5). Let $\{y_n\}$ be a sequence in S_1 and y a point in S_1. Suppose $\lim d_1^*(y_n, y) = 0$. Then $\lim d_1(y_n, y) = 0$ since $d_1(y_n, y) \leqq d_1^*(y_n, y)$. Conversely let $\lim d_1(y_n, y) = 0$. Then since $f: S_1 \to S_2$ is continuous, $\lim d_2(f(y_n), f(y)) = 0$. Thus $d_1^*(y_n, y)$ has a limit 0. Lastly $d_2(f(a), f(b)) \leqq d_1^*(a, b)$ and choosing $\delta = \epsilon$, it is clear that $f: S_1 \to S_2$ is uniformly continuous relative to d_1^* and d_2.

Let $f: S_1 \to S_2$ be continuous and d_1 and d_2 be metrics for S_1 and S_2 respectively. In general it is not possible to replace d_2 with an equivalent metric d_2^* such that $f: S_1 \to S_2$ is uniformly continuous relative to d_1 and d_2^*. Consider the following example. Let $S_1 \equiv \{x \mid x \text{ is real and } 0 \leqq x < 1 \text{ or } 1 < x \leqq 2\}$. Let $d_1(x, y) = |x - y|$ on S_1. Let $S_2 = \{a, b\}$ with $a \neq b$ and $d_2(a, b) = 1$. Let $f: S_1 \to S_2$ be defined by $f(x) = a$ if $0 \leqq x < 1$ and $= b$ if $1 < x \leqq 2$.

Clearly no equivalent metric d_2^* for d_2 will make $f: S_1 \to S$ uniformly continuous relative to d_1 and d_2^*.

Theorem 1 has an obvious extension to

THEOREM 2. *Let $f_i: S_1 \to S_2$ be continuous $(i = 1, \cdots, n)$ where S_1 and S_2 are metric spaces with metrics d_1 and d_2 respectively. Then there exists a metric d_1^* for S_1 which is equivalent to d_1 such that $f_i: S_1 \to S_2$ is uniformly continuous relative to d_1^* and d_2 for $i = 1, \cdots, n$.*

Proof. Let $d_1^*(a, b) = d_1(a, b) + d_2(f_1(a), f_1(b)) + \cdots + d_2(f_n(a), f_n(b))$. The remainder of the proof is almost identical to the proof for Theorem 1 and will be omitted.

We can get a stronger theorem by replacing d_1 and d_2 by equivalent metrics.

THEOREM 3. *Let f_i: $S_1 \to S_2$ be continuous for $i = 1, 2, \cdots$ where S_1 and S_2 are metric spaces with metrics d_1 and d_2 respectively. Then there exist metrics d_1^* and d_2^* for S_1 and S_2 respectively which are equivalent to d_1 and d_2 such that every f_i: $S_1 \to S_2$ is uniformly continuous relative to d_1^* and d_2^*.*

Proof. Let d_2^* be any equivalent metric for S_2 which is bounded by 1. It is well known that such exist. Let $d_1^*(a, b) = d_1(a, b) + \sum_{i=1}^{\infty} 2^{-i} d_2^*(f_i(a), f_i(b))$. It is clear that $d_1^*(a, b)$ is finite, nonnegative, and single valued on S_1; and that $d_1^*(a, b) = 0$ if and only if $a = b$. For if $d_1^*(a, b) = 0$, then $d_1(a, b) = 0$ and thus $a = b$. If $a = b$ then of course $d_1^*(a, b) = 0$. Clearly $d_1^*(a, b) = d_1^*(b, a)$. Now

$$
\begin{aligned}
&d_1^*(a, b) + d_1^*(b, c) \\
&\quad = d_1(a, b) + \sum_{i=1}^{\infty} 2^{-i} d_2^*(f_i(a), f_i(b)) + d_1(b, c) + \sum_{i=1}^{\infty} 2^{-i} d_2^*(f_i(b), f_i(c)) \\
&\quad = d_1(a, b) + d_1(b, c) + \sum_{i=1}^{\infty} 2^{-i} \{ d_2^*(f_i(a), f_i(b)) + d_2^*(f_i(b), f_i(c)) \} \\
&\quad \geqq d_1(a, c) + \sum_{i=1}^{\infty} 2^{-i} d_2^*(f_i(a), f_i(c)) = d_1^*(a, c).
\end{aligned}
$$

To show that d_1^* is equivalent to d_1, let (y_n) be a sequence in S_1 and y a point in S_1. Suppose $\lim d^*(y_n, y) = 0$. Then since $d_1(y_n, y) \leqq d_1^*(y_n, y)$, it follows that $\lim d_1(y_n, y) = 0$. Suppose then that $\lim d_1(y_n, y) = 0$. Then

$$
\begin{aligned}
d_1^*(y_n, y) &= d_1(y_n, y) + \sum_{i=1}^{\infty} 2^{-i} d_2^*(f_i(y_n), f_i(y)) \\
&\leqq d_1(y_n, y) + \sum_{i=1}^{N} 2^{-i} d_2^*(f_i(y_n), f_i(y)) + \epsilon/3 \text{ for some } N.
\end{aligned}
$$

Then

$$
\limsup d_1^*(y_n, y) \leqq \lim d_1(y_n, y) + \lim \sum_{i=1}^{N} 2^{-i} d_2^*(f_i(y_n), f_i(y)) + \epsilon/3 = \epsilon/3.
$$

Thus $\lim d_1^*(y_n, y) = 0$. Lastly $2^{-i} d_2^*(f_i(a), f_i(b)) \leqq d_1^*(a, b)$ or $d_2^*(f_i(a), f_i(b)) \leqq 2^i d_1^*(a, b)$. Hence for $\epsilon > 0$, choose $\delta = \epsilon/2^i$. This proves f_i: $S_1 \to S_2$ is uniformly continuous relative to d_1^* and d_2^*.

BOUNDS ON THE EXPECTATION FOR SUMS OF POWERS

J. A. NAVARRO,* General Electric Company, Advanced Electronics Center, Ithaca, N. Y.

In the article *Expectations for sums of powers* (this MONTHLY, vol. 66, 1959, pp. 494–496), Newman and Klamkin consider the following problem:

A sequence of independent random variables with a uniform distribution is chosen from the interval (0, 1). The process is continued until the sum of the nth powers of the chosen numbers exceeds 1. What is the expected number of such choices? They show that this expected number is given by

$$E_n = \sum_{i=0}^{\infty} [\Gamma(1 + 1/n)]^i/\Gamma(1 + i/n). \tag{1}$$

Further, for large n, $E_n \sim C \cdot n$ where

$$C = e^{e^{-\gamma}} - \int_{-\infty}^{\infty} \frac{e^{-e^u} du}{(u + \gamma)^2 + \pi^2}. \tag{2}$$

In this note upper and lower bounds on E_n are derived for all n, and hence a bound on the integral in (2) is obtained. Let

$$C_n = \frac{E_n}{n} = \frac{1}{n} \sum_{i=0}^{\infty} [\Gamma(1 + 1/n)]^i/\Gamma(1 + i/n);$$

then

$$C_n = \frac{1}{n} \sum_{i=0}^{\infty} \sum_{j=0}^{n-1} [\Gamma(1 + 1/n)]^{ni+j}/\Gamma(1+i+j/n).$$

Since $\Gamma(1+i) < \Gamma(1+i+j/n)$,

$$C_n < \frac{1}{n} \sum_{i=0}^{\infty} \sum_{j=0}^{n-1} [\Gamma(1 + 1/n)]^{ni+j}/\Gamma(1 + i) = \frac{e^{\Gamma^n(1+1/n)}[1 - \Gamma^n(1 + 1/n)]}{n[1 - \Gamma(1 + 1/n)]}.$$

Also $\Gamma(2+i) > \Gamma(1+i+j/n)$ and hence

$$C_n > \frac{1}{n} \sum_{i=0}^{\infty} \sum_{j=0}^{n-1} [\Gamma(1 + 1/n)]^{ni+j}/\Gamma(2 + i) = \frac{[e^{\Gamma^n(1+1/n)} - 1][1 - \Gamma^n(1 + 1/n)]}{n\Gamma^n(1 + 1/n)[1 - \Gamma(1 + 1/n)]}.$$

For large n, $\Gamma(1+1/n) \sim e^{-\gamma/n + O(1/n^2)}$, so that in the limit

$$\frac{(e^{e^{-\gamma}} - 1)(1 - e^{-\gamma})}{\gamma e^{-\gamma}} < C_\infty < \frac{e^{e^{-\gamma}}(1 - e^{-\gamma})}{\gamma},$$

or $1.0192 < C_\infty < 1.3320$, where γ is the Euler's constant.

* Now at IBM Research, Yorktown Heights, N. Y.

OSCILLATORY LIMITING BEHAVIOR OF A RANDOM SEQUENCE

P. M. ANSELONE AND P. PORCELLI, U. S. Army Mathematics Research Center, Madison, Wisconsin

Let X_n, $n=0, 1, \cdots$, be random variables defined in terms of a given sequence $\{(R_k, W_k): k \geqq 0\}$ of independent, identically distributed random vectors with nonnegative integer components by

$$X_n = R_k \quad \text{if} \quad \sum_{h=0}^{k-1} W_h \leqq n < \sum_{h=0}^{k} W_h.$$

For example, let the R_k be random choices of states of a system and the W_k random waits in these states. The system is in state i at time n if $X_n = i$.

The sequence $\{X_n: n \geqq 0\}$ is a discrete renewal process. For expositions of renewal theory see Smith [1] or Feller [2], Ch. 13. Let

$$P_{in} = \Pr\{X_n = i\}, \qquad p_i = \Pr\{R_k = i\},$$
$$a_{ij} = \Pr\{W_k = j \mid R_k = i\}, \qquad w_j = \Pr\{W_k = j\},$$

for i with $p_i > 0$ and $j = 0, 1, \cdots$. The P_{in} can be obtained by induction from the easily derived *renewal equation*,

$$P_{in} = \sum_{j=1}^{n} w_j P_{i,n-j} + p_i \sum_{j>n} a_{ij}.$$

The theory of Feller yields

$$\lim_{n\to\infty} P_{in} = \frac{p_i \mathrm{E}\{W_k \mid R_k = i\}}{\mathrm{E}\{W_k\}},$$

whenever the right member exists.

To treat cases of nonconvergence we define

$$a_i(x) = \sum_j a_{ij} x^j, \qquad w(x) = \sum_j w_j x^j, \qquad P_i(x) = \sum_n P_{in} x^n,$$

for $0 \leqq x \leqq 1$. By well-known properties of generating functions (*i.e.*, of Abel summability methods),

$$\liminf_{n\to\infty} P_{in} \leqq \liminf_{x\to 1} (1-x) P_i(x) \leqq \limsup_{x\to 1} (1-x) P_i(x) \leqq \limsup_{n\to\infty} P_{in}.$$

If $\lim_{n\to\infty} P_{in}$ exists then*

$$\lim_{n\to\infty} P_{in} = p_i \lim_{x\to 1} \frac{1-a_i(x)}{1-w(x)} = p_i \lim_{x\to 1} \frac{1-a_i(x)}{1-\sum_i p_i a_i(x)}.$$

If both $\lim_{n\to\infty} P_{in}$ and $\lim_{x\to\infty} a_i'(x)/w'(x)$ exist then

* Since the Abel method is stronger than any Cesàro method, we may replace $\lim_{n\to\infty} P_{in}$ here by any $C-m$ generalized limit. Cf [3], pp. 19–20. This observation is due to the referee.

$$\lim_{n\to\infty} P_{in} = p_i \lim_{x\to 1} \frac{a_i'(x)}{w'(x)} = \lim_{x\to 1} \frac{p_i a_i'(x)}{\sum_i p_i a_i'(x)}.$$

Consider the case with $p_1 = p_2 = \frac{1}{2}$. We have

$$p_1 \frac{1 - a_1(x)}{1 - w(x)} = \left[1 + \frac{1 - a_2(x)}{1 - a_1(x)}\right]^{-1}.$$

Choose α, β arbitrarily such that $0 \leqq \alpha \leqq \beta \leqq \infty$. There exist $a_i(x)$ such that

$$\liminf_{x\to 1} \frac{1 - a_2(x)}{1 - a_1(x)} = \alpha, \quad \limsup_{x\to 1} \frac{1 - a_2(x)}{1 - a_1(x)} = \beta.$$

Here is the idea of a proof of an even stronger assertion. Neglecting zero terms we write $a_i(x) = \sum_{j=0}^{\infty} c_{ij} x^{n_{ij}}$, where the c_{ij} are arbitrary such that $c_{ij} > 0$ and $a_i(1) = 1$, and the increasing sequences $\{n_{ij} : j \geqq 0\}$ of nonnegative integers are at our disposal. Let $r_{ik} = \sum_{j=m_{ik}}^{\infty} c_{ij}$, where the m_{ik} are chosen such that $m_{ik} < m_{i,k+1}$, $\lim_{k\to\infty} r_{2k}/r_{1k} = \beta$ and $\lim_{k\to\infty} r_{2,k+1}/r_{1k} = \alpha$. There exist n_{ij} and x_{ik} such that $x_{ik} \to 1$ and

$$\lim_{k\to\infty} \left[\frac{1 - a_2(x_{2k})}{1 - a_1(x_{2k})} - \frac{r_{2k}}{r_{1k}}\right] = 0, \qquad \lim_{k\to\infty} \left[\frac{1 - a_2(x_{1k})}{1 - a_1(x_{1k})} - \frac{r_{2,k+1}}{r_{1k}}\right] = 0.$$

Further details are omitted. If $\alpha = 0$ and $\beta = \infty$ we have $\liminf_{n\to\infty} P_{1n} = 0$ and $\limsup_{n\to\infty} P_{1n} = 1$; moreover, $\{P_{1n} : n \geqq 0\}$ does not converge in any Cesàro sense.

By similar methods the n_{ij} can be chosen such that

$$\lim_{x\to 1} [1 - a_2(x)]/[1 - a_1(x)]$$

exists but $\lim_{x\to 1} [a_2'(x)/a_1'(x)]$ does not exist, or, alternatively, such that neither limit exists. The basic principle involved in all of the proofs is that the relative rates of convergence of two series are affected by the insertion of gaps.

References

1. W. L. Smith, Renewal theory and its ramifications, J. Roy. Statist. Soc., Ser. B, vol. 20, No. 2, 1958.
2. W. Feller, Probability Theory, vol. 1, 2nd ed., New York, 1958.
3. O. Szász and J. Barlaz, Introduction to the Theory of Divergent Series, New York, 1948.

ON THE EVALUATION OF $\int_0^{\infty} (\sin^m t)/t^m dt$ BY THE TRAPEZOIDAL RULE

R. Butler, Manchester College of Science and Technology, England

The application of a formula due to Poisson shows that the integral $\int_0^{\infty} (\sin^m t)/t^m dt$ may be evaluated exactly by the trapezoidal rule using certain ranges of interval, dependent on m. Expansion of the series of terms thus formed

in Bernoulli polynomials enables the sum to be established, leading to a unified set of results involving powers of m and certain binomial coefficients.

One of these relations has not been found elsewhere in the literature by the author and is, therefore, believed to be a new result. Another of the relations gives an expression for the value of the integral, which takes a particularly simple form for $m \leqq 4$.

The formula of Poisson ([1], p. 443)

$$\sqrt{\alpha}\left\{\frac{1}{2}f(0) + \sum_{n=1}^{\infty} f(n\alpha)\right\} = \sqrt{\beta}\left\{\frac{1}{2}g(0) + \sum_{n=1}^{\infty} g(n\beta)\right\}$$

where

$$\alpha > 0, \qquad \alpha\beta = 2\pi, \qquad g(\theta) = \sqrt{(2/\pi)}\int_0^{\infty} f(t)\cos\theta t\,dt$$

may be written, on substituting $\alpha = h$, $G(\theta) = \sqrt{(\frac{1}{2}\pi)}g(\theta)$, as

$$\int_0^{\infty} f(t)dt = h\left\{\frac{1}{2}f(0) + \sum_{n=1}^{\infty} f(nh)\right\} - 2\sum_{n=1}^{\infty} G\left(\frac{2\pi n}{h}\right). \tag{1}$$

This form has been used by Fettis [2] in certain cases where the Fourier cosine transform is small in value; for example, when $f(t)$ is a periodic even function. In such cases it follows that the trapezoidal rule will provide a means of obtaining a good approximation to the integral. For some functions, however, the Fourier cosine transform is zero for certain ranges of θ and then the integral is given exactly by the trapezoidal rule for a range of interval determined by the range of θ. Thus if we take $f(t) = (\sin^m t)/t^m$, where m is a positive integer, then it is known that ([3], Sec. 1.6, Nos. 1, 12) $G(\theta) = 0$ for $\theta \geqq m$ if $m > 1$ and for $\theta > 1$ if $m = 1$. Hence,

$$I_m = \int_0^{\infty} \frac{\sin^m t}{t^m}\,dt = h\left\{\frac{1}{2} + \sum_{r=1}^{\infty} \sin^m(rh)/(rh)^m\right\},$$

if $h < 2\pi$ if $m = 1$, $h \leqq 2\pi/m$ if $m > 1$. Writing $S_m(h) = \sum_{r=1}^{\infty} \sin^m(rh)/r^m$, we obtain

$$S_m(h) = I_m h^{m-1} - \tfrac{1}{2}h^m \qquad (h < 2\pi \text{ if } m = 1,\ h \leqq 2\pi/m \text{ if } m > 1). \tag{2}$$

To obtain an expression for $S_m(h)$, the trigonometrical powers are first expanded in terms of multiple angles by means of the identities

$$\sin^m\theta = \begin{cases} (-1)^k 2^{-m}\left\{\sum_{j=0}^{k} (-1)^j \binom{m}{j} \sin(m-2j)\theta\right\} & (m = 2k+1), \\ (-1)^k 2^{1-m}\left\{\sum_{j=0}^{k-1} (-1)^j \binom{m}{j} \cos(m-2j)\theta + \frac{1}{2}(-1)^k \binom{m}{k}\right\} & (m = 2k). \end{cases} \tag{3}$$

If m is odd, there result $\frac{1}{2}(m+1)$ infinite series of the form $\sum_{r=1}^{\infty} \sin r\theta/r^m$ and,

if m is even, $\frac{1}{2}m$ infinite series of the form $\sum_{r=1}^{\infty} \cos r\theta/r^m$ together with the series $\sum_{r=1}^{\infty} 1/r^m$. In both cases the required combinations of these series involve binomial coefficient multipliers according to (3). Sums of the above forms are given by [4], p. 370, Nos. 5.5, 5.6 in terms of Bernoulli polynomials* and the further substitution of these results into the expansions gives

$$(4)\quad S_m(h) = \begin{cases} -(\pi^m/m!)\left[\sum_{j=0}^{k}(-1)^j\binom{m}{j}\phi_m\left\{\frac{(m-2j)h}{2\pi}\right\}\right] & (m = 2k+1), \\ -(\pi^m/m!)\left[\sum_{j=0}^{k-1}(-1)^j\binom{m}{j}\phi_m\left\{\frac{(m-2j)h}{2\pi}\right\}\right] & (m = 2k), \end{cases}$$

provided $0<h<2\pi$ if $m=1$ and $0\leqq h\leqq 2\pi/m$ if $m>1$ [4]. In the latter case, there is also a constant term

$$\frac{(-1)^k\pi^m B_k}{m!}\left\{\sum_{j=0}^{k-1}(-1)^j\binom{m}{j} + \frac{1}{2}(-1)^k\binom{m}{k}\right\},$$

but this is zero on summing the terms in the braces.

In (4) the Bernoulli polynomials are all of order m and so each will involve the powers h^m, h^{m-1}, h^{m-2}, h^{m-4}, h^{m-6}, $\cdots$, as far as h if m is odd or $m=2$ and otherwise as far as h^2 [4]. Collecting these terms we obtain

$$\text{coefficient of } h^m = -\{1/(2^m m!)\}\left\{m^m - \binom{m}{1}(m-2)^m + \cdots\right\} = a_m,$$

$$\text{coefficient of } h^{m-1} = \{\pi/(2^m(m-1)!)\}\left\{m^{m-1} - \binom{m}{1}(m-2)^{m-1} + \cdots\right\}$$

$$= a_{m-1},$$

$$\text{coefficient of } h^{m-2} = \{(\pi^2 B_1)/(2^{m-1}(m-2)!)\}\left\{m^{m-2} - \binom{m}{1}(m-2)^{m-2} + \cdots\right\}$$

$$= a_{m-2},$$

and so on. Hence the expression for $S_m(h)$ may be written

$$S_m(h) = a_m h^m + a_{m-1}h^{m-1} + a_{m-2}h^{m-2} + a_{m-4}h^{m-4} + \cdots,$$

for $0<h<2\pi$ if $m=1$ and $0\leqq h\leqq 2\pi/m$ if $m>1$, the final term being in h if m is odd or $m=2$ and otherwise in h^2.

The ranges of h for which this expression and (4) hold are identical and so direct comparison of them is permissible. It follows that

$$a_m = -\tfrac{1}{2}, \qquad a_{m-1} = I_m, \qquad a_{m-2} = a_{m-4} = a_{m-6} = \cdots = 0$$

and hence, writing in full, we obtain for positive integral m

* The Bernoulli numbers and polynomials used here follow the definition and notation of [3]. The Eulerian numbers are those used in [5], *viz.*, $E_0=1$, $E_1=1$, $E_2=1$, $E_3=2$, $E_4=5$, etc.

$$m^m - \binom{m}{1}(m-2)^m + \binom{m}{2}(m-4)^m + \cdots = 2^{m-1}m!$$

$$(5)\quad m^{m-1} - \binom{m}{1}(m-2)^{m-1} + \binom{m}{2}(m-4)^{m-1} + \cdots = \{(2^m(m-1)!/\pi\}I_m$$

$$m^{m-2r} - \binom{m}{1}(m-2)^{m-2r} + \binom{m}{2}(m-4)^{m-2r} + \cdots = 0 \quad (r = 1, 2, \cdots),$$

there being $\frac{1}{2}(m+1)$ or $\frac{1}{2}m$ terms on the left-hand side according as m is odd or even.

An alternative and simpler form for the value of I_m is obtainable in the cases $m \leqq 4$, when $h = \frac{1}{2}$ is permissible in (2). We then have ([5], No. 162)

$$S_m(\tfrac{1}{2}\pi) = 1 + \frac{1}{(-3)^m} + \frac{1}{5^m} + \frac{1}{(-7)^m} + \cdots = \frac{E_{m-1}\pi^m}{(m-1)!2^{m+1}},$$

where E_{m-1} represents the appropriate Eulerian number. Accordingly, from (2),

$$(6)\qquad I_m = \frac{1}{4}\pi\left\{\frac{E_{m-1}}{(m-1)!} + 1\right\} \qquad (m \leqq 4).$$

Referring to (5), it is noted that the value of the integral agrees with that already given in [6], No. 1927, and that the remaining results may be derived more directly from (3).

Suppose $m = 2k+1$ is odd. Then differentiation of the expression for $\sin^m \theta$ followed by setting $\theta = 0$ gives $\sum_{j=0}^{k}(-1)^j\binom{m}{j}(m-2j) = 0$. Repeating this process and setting $\theta = 0$ after successive differentiations gives the last of (5) and, similarly, if $m = 2k$.

Furthermore, on differentiating either of (3) m times and setting $\theta = 0$, we obtain

$$\left[\frac{d^m}{d\theta^m}\sin^m\theta\right]_{\theta=0} = 2^{1-m}\left[m^m - \binom{m}{1}(m-2)^m + \cdots\right],$$

the number of terms being $\frac{1}{2}(m+1)$ or $\frac{1}{2}m$ according as m is odd or even. But the left-hand side is easily shown to be $m!$, whence the first of relations (5) is established again.

References

1. E. C. Titchmarsh, The Theory of Functions, 2nd ed., Oxford, 1952.
2. H. E. Fettis, Numerical calculation of certain definite integrals by Poisson's summation formula, Math. Tables Aids Comput., vol. 9, 1955, pp. 85–92.
3. A. Erdelyi, W. Magnus, F. Oberhettinger, and F. G. Tricome, Tables of Integral Transforms, vol. 1, New York, 1954.
4. T. J. I'A. Bromwich, Introduction to the Theory of Infinite Series, 2nd ed., New York, 1955.
5. L. B. W. Jolley, Summation of Series, London, 1925.
6. J. Wolstenholme, Mathematical Problems, 3rd ed., New York, 1891.

ISOLATED POINTS IN A BANACH SPACE

FRANK HAHN, University of Illinois

It is known* that any subset of the real line has at most a countable number of points that are isolated from either the right or the left. The theorem proved here is a generalization of this result to separable Banach spaces.

In the following B shall be a separable Banach space over the reals and $S'(x, \rho)$ will be the shell of the sphere of radius ρ about the point x. The set $X=\{x_i: i=1, 2, \cdots\}$ shall be a countable dense subset of $S'(0, 1)$.

DEFINITION 1. *A set $P\subset B$ is positive linear if it is contained in a positive half ray; that is, there exists an $x\in B$ such that $P\subset\{\lambda x: \lambda>0\}$.*

DEFINITION 2. *If $P\subset B$, then a point x in P is isolated from the left if there is a number $\epsilon>0$ such that if $0<\lambda<1$ and λx is in P then $\|x-\lambda x\|\geqq\epsilon$.*

LEMMA. *If $P\subset B$ is positive linear then P contains at most a countable number of points isolated from the left.*

Proof. This follows from the fact that a positive half ray is homeomorphic to $(0, +\infty)$.

THEOREM. *If $M\subset B$ then for each $\rho>0$, with at most a countable number of exceptions, we have*

$$S(0, \rho)\cap M - S^0(0, \rho)\cap M = \varnothing. \tag{1}$$

Proof. Let $\mathfrak{N}$ be the collection of all spheres about the origin for which (1) does not hold. The theorem shall be demonstrated by constructing a 1:1 mapping θ of $\mathfrak{N}$ into a countable set I.

If $y\in S'(0, 1)$ and $a>0$ let $N(y, a)=S(y, a)\cap S'(0, 1)$ and if $x\in N(y, a)$ let $M(x, y, a)=\{\lambda y: \lambda x\in M, \lambda>0\}$. Let $M(y, a)=\bigcup_{x\in N(y, a)} M(x, y, a)$. Notice that if $x\in N(y, a)$ and if λx and $\mu x\in M$ then λy and $\mu y\in M(x, y, a)$ and $\|\lambda x-\mu x\|=\|\lambda y-\mu y\|$.

Since $M(y, a)$ is positive linear, let $I(y, a)$ be the countable set of points of $M(y, a)$ which are isolated from the left. If $i, n=1, 2, \cdots$ then $I=\bigcup_{i,n} I(x_i, 1/n)$ is a countable set.

If $S(0, \rho)\in\mathfrak{N}$ then there is a point y such that $\|y\|=1$, $\rho y\in M$, and there is an $\epsilon>0$ such that $\rho>\epsilon>0$ and

$$S^0(0, \rho)\cap M\cap S(\rho y, \epsilon) = \varnothing. \tag{2}$$

Choose n and $x_i\in X$ such that $\|\rho x_i-\rho y\|<\rho/n<\epsilon/4$ and let $\theta(S(0, \rho))=\rho x_i$.

Since $\rho y\in S(\rho x_i, \rho/n)$ and $\rho y\in M$ we see that $y\in S(x_i, 1/n)\cap S'(0, 1)=N(x_i, 1/n)$ and consequently $\rho x_i\in M(x_i, 1/n)$. From the fact that $S(\rho x_i, \rho/n)\subset S(\rho y, \epsilon/2)$ we see that $S(x_i, 1/n)\subset S(y, \epsilon/2\rho)$, $N(x_i, 1/n)\subset N(y, \epsilon/2\rho)$. In order to show that $\rho x_i\in I$ it suffices to show that ρx_i is isolated from the left in

* S. Saks, Theory of the Integral, New York, 1937, p. 206.

$M(x_i, 1/n)$. If $0<\lambda<1$ and $\lambda\rho x_i \in M(x_i, 1/n)$ then there is a $z \in N(x_i, 1/n) \subset N(y, \epsilon/2\rho)$ such that $\lambda\rho z \in M$. Since $z \in S(y, \epsilon/2\rho)$ we have

$$\rho z \in S(\rho y, \epsilon/2). \tag{3}$$

Since $\|z\| = 1$ and $\lambda < 1$ we have $\lambda\rho z \in S^0(0, \rho) \cap M$ and from (2) it follows that $\lambda\rho z \notin S(\rho y, \epsilon)$. Combining this and (3) we obtain $\|\lambda\rho x_i - \rho x_i\| = \|\lambda\rho z - \rho z\| \geqq \epsilon/2$ and it follows that $\rho x_i \in I$.

If $S(0, \rho) \neq S(0, \beta)$ then $\rho \neq \beta$ and $\theta(S(0, \rho)) = \rho x_i \neq \beta x_j = \theta(S(0, \beta))$ from which it follows that θ is 1:1.

COROLLARY. *If $M \subset B$ and x is fixed then for each $\rho > 0$, with at most a countable number of exceptions, we have*

$$S(x, \rho) \cap M - \overline{S^0(x, \rho) \cap M} = \varnothing.$$

Proof. This follows from the fact that translation by $-x$ is a homeomorphism.

The hypothesis that B is separable cannot be dropped. If B is a Hilbert space over the reals with an orthonormal basis $\{\phi_c : c \in [0, 1]\}$ and if $M = \{c\phi_c : c \in [0, 1]\}$ then the theorem is false.

AN IDENTITY OF CAYLEY

L. CARLITZ, Duke University AND JACK LEVINE, North Carolina State College

Cayley [1] proved that

$$\begin{vmatrix} a_1^{-1} & a_2^{-1} & a_3^{-1} \\ b_1^{-1} & b_2^{-1} & b_3^{-1} \\ c_1^{-1} & c_2^{-1} & c_3^{-1} \end{vmatrix}^{+} \cdot \begin{vmatrix} a_1^{-1} & a_2^{-1} & a_3^{-1} \\ b_1^{-1} & b_2^{-1} & b_3^{-1} \\ c_1^{-1} & c_2^{-1} & c_3^{-1} \end{vmatrix}^{+} = \begin{vmatrix} a_1^{-2} & a_2^{-2} & a_3^{-2} \\ b_1^{-2} & b_2^{-2} & b_3^{-2} \\ c_1^{-2} & c_2^{-2} & c_3^{-2} \end{vmatrix}$$

provided

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

In this note we prove the following generalization.

THEOREM. *If the matrix (a_{rs}) is of order n and of rank $\leqq 2$, and all $a_{rs} \neq 0$, then*

$$\left| a_{ij}^{-1} \right|^{+} \cdot \left| a_{ij}^{-1} \right| = \left| a_{ij}^{-2} \right|. \tag{1}$$

The case $n = 3$ is Cayley's result; the case $n = 4$ has been treated by Levine in a recent paper [2]. The proof in the general case is similar to Levine's proof. By a formula of Muir [3]

$$(2)\qquad \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}_{+} \cdot \begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}_{+} = \sum \pm \begin{vmatrix} a_{11}x_{i1} & \cdots & a_{1n}x_{in} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1}x_{k1} & \cdots & a_{nn}x_{kn} \end{vmatrix},$$

where the summation is extended over all $n!$ permutations $ij \cdots k$ of $12 \cdots n$ and the $+$ or $-$ sign is used according as the permutation is even or odd.

Now in (2) take $x_{rs}=a_{rs}$ and then replace a_{rs} by a_{rs}^{-1}. Then the term on the right corresponding to the identity permutation evidently yields the right member of (1). Consider any other permutation p. Write p as a product of disjoint cycles. If any one of the cycles is a transposition it follows that the corresponding determinant in the right member of (2) has two equal rows and therefore vanishes. For a cycle of length h, where $h \geqq 3$, the corresponding determinant also vanishes.

We may assume that the rows of the matrix (a_{rs}) are linearly dependent on the first two rows and that the cycle is $(12 \cdots h)$, where $n \geqq h \geqq 3$. The first h rows of the corresponding determinant are

$$\begin{bmatrix} a_{11}^{-1} & a_{21}^{-1} & \cdots & a_{1n}^{-1} & a_{2n}^{-1} \\ a_{21}^{-1} & a_{31}^{-1} & \cdots & a_{2n}^{-1} & a_{3n}^{-1} \\ \cdot & & & \cdot & \\ \cdot & & & \cdot & \\ \cdot & & & \cdot & \\ a_{h1}^{-1} & a_{11}^{-1} & \cdots & a_{hn}^{-1} & a_{1n}^{-1} \end{bmatrix} = (a_{ij}^{-1}\, a_{i'j}^{-1}),$$

where $i'=i+1$ for $1 \leqq i < h$, $i'=1$ for $i=h$. Then

$$(a_{ij}^{-1}\, a_{i'j}^{-1}) = \left(k_j \prod_{\substack{s=1 \\ s \neq i,\, i'}}^{h} a_{sj} \right),$$

where $k_j = \prod_{s=1}^{h} a_{sj}^{-1}$ $(j=1, \cdots, n)$. Since $k_j \neq 0$ this latter matrix has the same rank as the matrix

$$A = \left(\prod_{\substack{s=1 \\ s \neq i,\, i'}}^{h} a_{sj} \right) \qquad (i = 1, \cdots, h;\ j = 1, \cdots, n).$$

By our assumption concerning (a_{rs}), we have

$$a_{sj} = b_s a_{1j} + c_s a_{2j} \qquad (s.\ j = 1, \cdots, n).$$

Hence

$$\prod_{\substack{s=1 \\ s \neq i,\, i'}}^{h} a_{sj} \prod_{s=1}^{h} (b_s a_{1j} + c_s a_{2j}) = \sum_{s=0}^{h-2} d_{is} a_{1j}^{h-2-s} a_{2j}^{s},$$

so that the row space of A is spanned by $h-1$ vectors and the rank of A is less

than h. Thus the corresponding determinant vanishes. This completes the proof of the theorem.

For a cycle of even length it is easily seen that the corresponding term in (2) is cancelled by the determinant in which this cycle is replaced by its inverse; this does not make use of the hypothesis concerning the rank of (a_{rs}). Consequently we may state the following corollary of Muir's formula (2):

If (a_{rs}) is an arbitrary matrix of order n then

$$\begin{vmatrix} + & & + \\ a_{11} & \cdots & a_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \sum \begin{vmatrix} a_{11}a_{i1} & \cdots & a_{1n}a_{in} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1}a_{k1} & \cdots & a_{nn}a_{kn} \end{vmatrix},$$

where the summation is extended over these permutations $(ij \cdots k)$ that are products of disjoint cycles of odd length.

References

1. A. Cayley, Note sur les normales d'une conique, J. Reine Angew. Math., vol. 56, 1858, pp. 182–185 (Collected Papers, vol. 4, pp. 74–77).

2. Jack Levine, Note on an identity of Cayley, this MONTHLY, vol. 66, 1959, pp. 289–291.

3. T. Muir, On a class of permanent symmetric functions, Proc. Roy. Soc. Edinburgh, vol. 11, 1882, pp. 409–418.

ANOTHER PROOF OF THE FOUR VERTEX THEOREM

LESTER E. DUBINS, Institute for Advanced Study

The fundamental theorem of Schur [2, 3] has a number of fruitful applications. It seems to be of interest to observe that the four vertex theorem for convex curves is also an easy consequence of Schur's theorem. The latter states: *Let D be a plane arc which together with the chord connecting its end points forms a convex curve. Let D^* be an arc of the same length as D and let s be the arc length parameter for both curves. Suppose that at corresponding points the continuous curvatures $k(s)$ and $k^*(s)$ satisfy $k(s) \geqq k^*(s)$ and that d and d^* denote the lengths of the chords joining the end points of D and D^* respectively. Then $d \leqq d^*$, and equality holds only if for all s, $k(s) = k^*(s)$.*

FOUR VERTEX THEOREM. *Let C be any closed convex curve in the plane with continuous curvature. Then there are at least four points on C at which the curvature assumes a relative maximum or minimum.*

It is a mild convenience in the proof of the theorem to assume that the length of C is 2π and that C is oriented so that the curvature is everywhere nonnegative. Then either C is a circle, in which event there is nothing to prove, or there are points on C where the curvature is strictly greater than the average and points where it is strictly less than average. Thus, the set D_1 of points where the curvature is greater than or equal to 1 is a nonempty proper closed subset of C. We claim that it is not an arc. Suppose that it were; then the end

points of D_1 also determine an arc, D_2^*, the complementary arc to D_1. Now let E be a circle of length 2π, and let D_1^* be an arc of E of the same length as D_1 and let D_2 be the complementary arc to D_1^*. Then, by Schur's lemma, $d_1 < d_1^*$ and $d_2 < d_2^*$. But $d_1 = d_2^*$ and $d_1^* = d_2$. This contradiction shows that D_1 is indeed not an arc. Since D_1 is disconnected, there are two points on C, each not in D_1, which cut C into two arcs, each of which contain points of D_1. Hence the maximum value of the curvature on each of these arcs is assumed at interior points of the arcs. At each of these points the curvature has a relative maximum. The usual observation that between any two relative maxima there exists a relative minimum completes the proof of the theorem.

The idea of the above proof also establishes another characterization of the circle. Namely,

THEOREM 1. *Let C be a closed convex planar curve with continuous curvature. Suppose that there is an arc of C which contains every point where the curvature is strictly greater than the average curvature of C, but contains no point where it is strictly less than the average. Then C is a circle.*

References

1. W. Blaschke, Einführung in die Differentialgeometrie, Berlin/Göttingen/Heidelberg, 1950.
2. E. Schmidt, Über das Extremum der Bogenlänge einer Raumkurve bei vorgeschriebenen Einschränkungen ihrer Krümmung, S.-B. Akad. Wiss. Berlin, 1925, pp. 485–490.
3. A. Schur, Über die Schwarzsche Extremaleigenschaft des Kreises unter den Kurven konstanter Krümmung, Math. Ann. vol. 83, 1921, pp. 143–148.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

A PROOF OF THE VECTOR TRIPLE CROSS PRODUCT IDENTITY

ROBERT C. WREDE, San Jose State College

The purpose of this note is to present a relatively simple proof of the following relation for the triple cross product of three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b}). \tag{1}$$

This identity is usually left for the students to verify with the assumption that they will simply write out the two members of the equation in component form and then compare them. An exception to this normal procedure is made by Lass, *Vector and Tensor Analysis*. There he presents a rather nice proof in which he maintains the spirit of the subject by developing the relation in terms of the vector language.

Those students who have been exposed to a course in tensor analysis know

that the notation of that subject makes the proof of (1) almost trivial.* The following proof is an adaptation of that given in terms of the tensor notation but stated in such a way that a minimum number of new concepts need to be introduced.

DEFINITION 1. *Let the components of a vector* **A** *be represented by either* A_i *or* A^i, $i=1, 2, 3$. *Then*

$$A^{[i}B^{j]} = A^iB^j - A^jB^i, \qquad A_{[i}B_{j]} = A_iB_j - A_jB_i. \tag{2}$$

On the basis of (2) we can make the following correspondences

$$\mathbf{b}\times\mathbf{c} \cdots (b^{[j}c^{k]})_i \qquad i, j, k = 1, 2, 3. \tag{3}$$

That is the components of $\mathbf{b}\times\mathbf{c}$ are determined by those three representations of the right-hand member of (3) for which i, j, k are even permutations of 1, 2, 3. Each odd permutation of i, j, k determines the negative of a component of $\mathbf{b}\times\mathbf{c}$ and in the remaining situations, that is when any pair i, j, k take on the same one of the values 1, 2, 3, we define the expression to be zero. With the same understanding as with respect to (3) we have

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \cdots (a_{[p}(b^{[j}c^{k]})_{i]})_r. \tag{4}$$

The theorem to be proved may then be stated as follows:

THEOREM

$$(a_{[p}(b^{[j}c^{k]})_{i]})_r = b^r\sum_{s=1}^{3}a_sc^s - c^r\sum_{s=1}^{3}a_sb^s. \tag{5}$$

Proof. The three components of $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ are completely determined by the choice of r and the fact that r, p, i and i, j, k must each form an even permutation of 1, 2, 3. For example, if $r=1$ then $p=2$, $i=3$. Since $i=3$ it follows that $j=1$, $k=2$. Now suppose that α, β, γ is an even permutation of 1, 2, 3 then if $r=\alpha$ the right number of (4) has the following form

$$(a_{[\beta}(b^{[\alpha}c^{\beta]})_{\gamma]})_\alpha. \tag{6}$$

Careful examination of (6) produces the following two facts:

The components of $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ are such that

(a) the indices corresponding to r and j agree,

(b) the indices corresponding to p and k agree.

With (6) in mind we may replace the component representation of $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ by

$$\sum_{s=1}^{3}a_{[s}(b^{[r}c^{s]})_{i]}, \tag{7}$$

* Nathaniel Coburn, Vector and Tensor Analysis, New York, 1955, p. 120.

where r has replaced j and been dropped as unnecessary in its original position. Moreover p and k have been replaced by a summation index s. The summation does not introduce any new nonzero terms because of the understanding with respect to the representations (3) and (4). For example if $r=1$ then a component of $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ corresponds to $i=3$ and hence $s=2$. In this instance $s=1$ or 3 produces a zero term.

By definition, (7) can be written

$$\sum_{s=1}^{3} a_s(b^{[r}c^{s]})_i - \sum_{s=1}^{3} a_i(b^{[r}c^{s]})_s. \tag{8}$$

Since the same index appears twice in the expression $(b^{[r}c^{s]})_s$ the right-hand term of (8) is equal to zero. Hence (8) reduces to

$$\sum_{s=1}^{3} a_s(b^{[r}c^{s]})_i. \tag{9}$$

Now in (9) given r and s the value of i such that r, s, i forms an even permutation of 1, 2, 3 is uniquely determined. Hence the subscript i may be dropped and (9) written in the form

$$b^r \sum_{s=1}^{3} a_s c^s - c^r \sum_{s=1}^{3} a_s b^s.$$

This completes the proof.

ON THE IRRATIONALITY OF ROOTS

HERBERT E. VAUGHAN, University of Illinois

THEOREM. *For each integer $m\geqq 2$, there is no positive integer whose mth root is a nonintegral rational number.*

Briefly, roots of positive integers are irrational unless they are integral. This theorem is well known and is an easy consequence of, for example, the unique factorization theorem. The purpose of this note is to call attention to a proof of the nonexistence of nonintegral rational roots of integers which depends neither on the unique factorization theorem nor on any theorem concerning divisibility properties peculiar to integers. The proof is by mathematical induction on m. The first step is the proof that

(1) *there is no positive integer whose square root is a nonintegral rational number.*

This part of the proof can be given, after some preparation, to students who are unacquainted with mathematical induction but have had some experience with inequations. [For example, it can be given near the end of a modern ninth-grade mathematics course.] Aside from the familiar ring-properties of the integers and some elementary theorems on inequations, all that is needed for the proof are the two intuitively acceptable theorems:

(A) *Each nonempty set of positive integers contains a least member;*

(B) *For each nonintegral rational number r, there is an integer k such that $k<r<k+1$.*

Proof of (1). Suppose that n is a positive integer such that $\sqrt{n}$ is a nonintegral rational number. Then, by (A), there is a least positive integer q such that $\sqrt{n}\cdot q$ is an integer, and, by (B), there is an integer k such that $k<\sqrt{n}<k+1$. It follows, from simple theorems on inequations, that, since $q>0$, $0<\sqrt{n}\cdot q-kq<q$. Since, consequently, $\sqrt{n}\cdot q-kq$ is a positive integer smaller than q, $\sqrt{n}(\sqrt{n}\cdot q-kq)$ is not an integer. However, $\sqrt{n}(\sqrt{n}\cdot q-kq)=nq-k(\sqrt{n}\cdot q)$, an integer. This contradiction shows that there is no positive integer whose square root is a nonintegral rational number.

The second step is the proof that

(2) *for each integer $m\geqq 2$, if there is no positive integer whose mth root is a nonintegral rational number then there is no positive integer whose $(m+1)$st is a nonintegral rational number.*

In this part of the proof we use the theorem:

(C) *For each choice of integers m, i, and j, there is an integer N such that $i^m-j^m=(i-j)N$.*

[The case $j=0$ is obvious and the case $j\neq 0$ (the only one needed here) is an easy consequence of the theorem according to which, for each positive integer m and each real number x, $x^m-1=(x-1)\sum_{v=0}^{m-1}x^v$. Students who are taking a modern eleventh-grade mathematics course should be acquainted with this theorem on the sum of a geometric progression, as well as with the method of mathematical induction by which it is proved. The whole content of this note, suitably expanded, makes appropriate material for such students, and furnishes them an interesting application of mathematical induction.]

Proof of (2). Suppose that there is no positive integer whose *mth* root is a nonintegral rational number. It follows that there is no nonintegral rational number whose mth power is a positive integer. Suppose, now, that n is a positive integer such that $\sqrt[m+1]{n}$ is a nonintegral rational number. Then, by (A), there is a least positive integer q such that $\sqrt[m+1]{n}\cdot q$ is an integer, say p, and, by (B), there is an integer k such that $k<p/q<k+1$. It follows [as in the first part of the proof] that $p-kq$ is a positive integer smaller than q, so $\sqrt[m+1]{n}(p-kq)$ is not an integer. Hence, $(p/q)(p-kq)$ is a nonintegral rational number and, as remarked above, it follows [from the "inductive hypothesis"] that

$$[(p/q)(p-kq)]^m,$$

although certainly positive, is not a positive integer. But, by (C), there is an integer N such that $(p-kq)^m-(-kq)^m=pN$. Hence, $[(p/q)(p-kq)]^m=(p^m/q^m)(pN)+(-kq)^m=nqN+(-kp)^m$, an integer. This contradiction shows

that, assuming that there is no positive integer whose mth root is a nonintegral rational number, there is no positive integer whose $(m+1)st$ root is a nonintegral rational number.

The theorem now follows from (1) and (2), by mathematical induction.

It is perhaps interesting to note that the proof of (2) also proves that

(2′) *for each positive integer m, if there is no positive integer whose mth root is a nonintegral rational number then there is no positive integer whose $(m+1)st$ root is a nonintegral rational number.*

Now, it is trivial that

(1′) *there is no positive integer whose 1st root is a nonintegral rational number,*

and, from (1′) and (2′) one obtains, by mathematical induction, the theorem:

For each positive integer m, there is no positive integer whose mth root is a nonintegral rational number.

Although the proof just outlined is somewhat shorter than that set forth above, the latter has the advantage of showing how little knowledge is required for an understanding of the proof of (1).

THE FUNDAMENTAL THEOREM OF ALGEBRA

J. E. Eaton, Queens College

The following proof of the Fundamental Theorem of Algebra has three advantages. It uses methods and concepts which are natural to the algebra courses where the proof is frequently given. Its underlying idea is intuitively clear to the undergraduate. It is easy to follow and not unduly long, once the preliminary spade work is done.

If a polynomial with real coefficients is transformed by dividing all of its roots by the absolute value of one of them, the resulting polynomial will have a root on the unit circle. It will then have a factor in common with the polynomial formed from the reciprocals of its roots and the resultant of the two should vanish. The essence of the proof is to show that for some value of the divisor the resultant does indeed vanish.

The argument is restricted to the real field. For this reason the theorem has been phrased as the factorization of real polynomials. Since the product of a polynomial with complex coefficients and its conjugate is real, the extension to the usual form of the fundamental theorem is immediate.

Theorem. *Any polynomial with real coefficients may be expressed as a product of linear and quadratic polynomials with real coefficients.*

Proof. The proof is by induction on n, the degree of the polynomial. Write the polynomial as

$$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$$

and suppose that neither a_0 nor a_n is zero. Consider the polynomials

$$P_1(x) = a_0r^nx^n + a_1r^{n-1}x^{n-1} + \cdots + a_n,$$

$$P_2(x) = a_nx^n + a_{n-1}rx^{n-1} + \cdots + a_0r^n.$$

Bézout's form of the resultant* of $P_1(x)$ and $P_2(x)$ is a determinant of order n symmetric with respect to its main diagonal and also, in this instance, symmetric with respect to its other diagonal. Denote by D the determinant obtained from the resultant by interchanging the diagonals through successive interchanges of rows. Then D vanishes when $P_1(x)$ and $P_2(x)$ have a common factor. Moreover, D is a symmetric determinant with the following properties. Every element on the main diagonal is a polynomial in r whose leading and constant terms are respectively $a_0^2r^{2n}$ and $-a_n^2$. Every element not on the main diagonal is a polynomial in r of degree less than $2n$ whose constant term is 0. Let r_0 be the largest value of r for which any principal minor of D vanishes. Such a number exists since every principal minor of odd order is negative when $r=0$ and positive for sufficiently large values of r. If D is not zero for $r=r_0$, there is a nest of principal minors of consecutive orders, say D_k, D_{k+1}, D_{k+2}, such that only $D_{k+1}=0$ when $r=r_0$. The existence of this nest of minors is a general property of nonsingular symmetric matrices that contain at least one principal minor that is zero. It is a consequence of the fact that not all the principal minors of two consecutive orders can vanish† and may be proved by a simple induction on the order of the matrix. It is necessary, however, to define a zero-order minor to be $+1$. Since $D_{k+1}=0$, it is well known that D_k and D_{k+2} have opposite signs.‡ As they are both positive for sufficiently large values of r, one of them becomes zero for some $r>r_0$, a contradiction. Hence D vanishes for $r=r_0$. This implies that there are relatively prime polynomials $Q_1(x)$ and $Q_2(x)$ of degree less than n such that for $r=r_0$

$$P_1(x)Q_1(x) = P_2(x)Q_2(x).$$

If $Q_2(x)$ is not a constant, the induction assumption insures that it is a product of linear and quadratic factors which must then divide $P_1(x)$. The quotient is a similar product and thus so also is $P_1(x)$. The factorization of $P(x)$ is obtained by replacing x by x/r_0. If $Q_2(x)$ is a constant, $P_1(x)$ for $r=r_0$ is a reciprocal polynomial. Either $x\pm1$ is a factor and the previous argument applies or $P_1(x)$ may be transformed to a polynomial of half the degree. Therefore $P_1(x)$ is a product of quadratic and quartic reciprocal polynomials. But the factorization of quartic reciprocal polynomials is readily established.

---

* W. S. Burnside and A. W. Panton, The Theory of Equations, vol. II, Dublin, 1904, pp. 76–78.

† Maxime Bôcher, Introduction to Higher Algebra, New York, 1935, p. 57.

‡ W. S. Burnside and A. W. Panton, op. cit., p. 50. The restriction to leading minors imposed by Burnside and Panton is easily removed.

THE EXTENDED LAW OF THE MEAN BY A TRANSLATION-ROTATION OF AXES

JACQUELINE P. EVANS, Wellesley College

In the Classroom Note *Proof of the mean value theorem* (this MONTHLY, vol. 65, 1958, pp. 362–364), C. L. Wang uses a translation-rotation of the Cartesian coordinate system to obtain the Mean Value Theorem from Rolle's Theorem. The chief merit of this proof is that it provides a natural geometric motivation for the Mean Value Theorem. However, as the author points out, there are cases where his method of proof breaks down. This occurs whenever the graph of the given curve doubles back upon itself and is not the graph of a *function* in terms of the XY-coordinate system. (See Fig. 1 for example.)

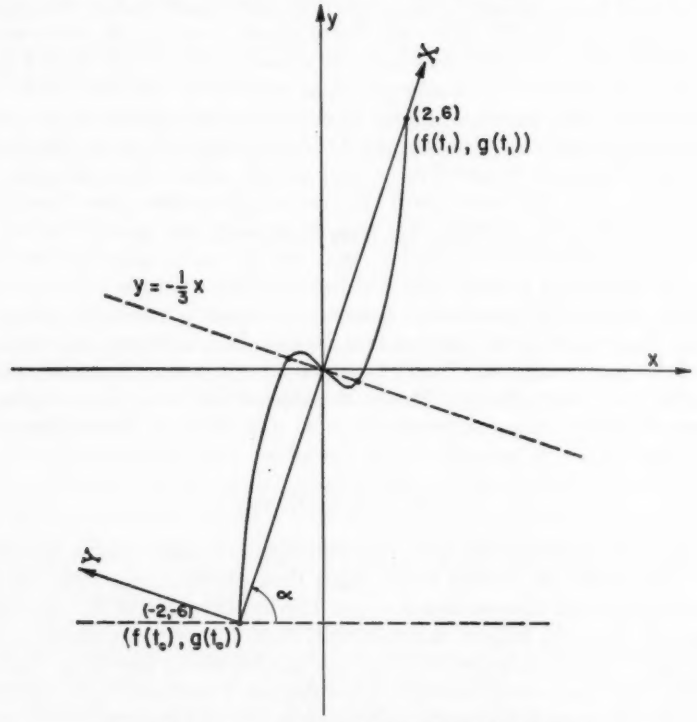


FIG. 1. $y=x^3-x$, $-2\leq x\leq 2$; $x=f(t)$, $y=g(t)$, $t_0\leq t\leq t_1$.

The method of proof used by Wang can be adapted to cover all cases. When this is done we obtain the Extended Law of the Mean. The proof retains the virtue of yielding a natural motivation and interpretation of this theorem and avoids the use of artificially concocted functions to which Rolle's Theorem is usually applied.

Let $x=f(t)$, $y=g(t)$, $t_0 \leqq t \leqq t_1$, where f and g are continuous for $t_0 \leqq t \leqq t_1$, and $f'(t)$, $g'(t)$ exist for $t_0<t<t_1$. If $g(t_0)=g(t_1)=0$, there exists, by Rolle's Theorem applied to g, at least one T, $t_0<T<t_1$, such that $g'(T)=0$. Let the given equations be regarded as the parametric representation of a given curve. If $f'(T)\neq 0$, this curve must have a horizontal tangent. If $f'(T)=0$, we can draw no conclusion without further investigation.

Now assume that $g(t_0)\neq g(t_1)$ but retain the other conditions on f and g. The geometric situation described above can be shifted to a new coordinate system by a translation and rotation of the xy-system into the XY-system. Using the results of Wang, the parametric equations of the given curve, in terms of the new coordinate system, are:

$$F(t) = X = [g(t) - g(t_0)] \sin \alpha + [f(t) - f(t_0)] \cos \alpha,$$
$$G(t) = Y = [g(t) - g(t_0)] \cos \alpha - [f(t) - f(t_0)] \sin \alpha, \qquad t_0 \leqq t \leqq t_1.$$

By Rolle's Theorem applied to G there exists at least one T, $t_0<T<t_1$, such that $G'(T) = g'(T) \cos \alpha - f'(T) \sin \alpha = 0$, from which it follows that $g'(T)[f(t_1)-f(t_0)]=f'(T)[g(t_1)-g(t_0)]$. This result when interpreted geometrically in terms of the xy-coordinate system simply says that either there is one point on the given curve such that the tangent there is parallel to the secant through the two endpoints of the curve or else there is an indeterminate point. If we make the added assumption that $f'(t)\neq 0$, $t_0<t<t_1$, the first case occurs and we can write:

$$\frac{g'(T)}{f'(T)} = \frac{g(t_1) - g(t_0)}{f(t_1) - f(t_0)},$$

thus obtaining the Extended Law of the Mean.

A NOTE ON TWO CONVERGENCE TESTS

JAMES MICHELOW, University of Washington and University of Chile

In the theory of series of positive terms we use four elementary convergence tests that depend on the following limits;

I. $\lim_{n\to\infty} \sqrt{u_n} = \lambda_1$ (Cauchy's first test),

II. $\lim_{n\to\infty} (u_{n+1}/u_n) = \lambda_2$ (D'Alembert's test),

III. $\lim_{n\to\infty} \dfrac{\ln (1/u_n)}{\ln (n)} = \lambda_3$ (Cauchy's second test [1])

IV. $\lim_{n\to\infty} [n\{1 - (u_{n+1}/u_n)\}] = \lambda_4$ (Raabe-Duhamel's test [2]),

where u_n is the term of order n in the series and ln denotes the natural logarithm.

It is a known result that if λ_2 exists, then λ_1 exists. Moreover, $\lambda_1=\lambda_2$. The

two tests are then equivalent, the first being more general than the second.

The purpose of this note is to establish a similar relationship between the last two tests. This relationship is embodied in the following:

THEOREM. *If* λ_4 *exists, finite or infinite, then* λ_3 *exists and* $\lambda_3=\lambda_4$.

Proof. We shall use the following theorem:

If $\lim_{n\to\infty}\{(a_{n+1}-a_n)/(b_{n+1}-b_n)\}=\mu$, *where* $b_{n+1}>b_n$ *and* $\lim_{n\to\infty} b_n=\infty$, *then* $\lim_{n\to\infty}(a_n/b_n)=\mu$. ($\mu$ *can be finite or infinite* [3].)

Let us put $a_n=\ln(1/u_n)$, $b_n=\ln(n)$. Then

$$\lim_{n\to\infty}\frac{\ln(1/u_{n+1})-\ln(1/u_n)}{\ln(n+1)-\ln(n)}=\lim_{n\to\infty}[-n\ln(u_{n+1}/u_n)].$$

By the theorem quoted above, if $\lim_{n\to\infty}[-n\ln(u_{n+1}/u_n)]=\mu$, then

$$\lim_{n\to\infty}\frac{\ln(1/u_n)}{\ln(n)}=\mu.$$

But $\ln(1+x)=x-\frac{1}{2}x^2M(x)$, where $M(x)>0$ and is bounded in $-1<B\leqq x\leqq A$. Now let us assume that λ_4 is finite; then

$$\lim_{n\to\infty}(u_{n+1}/u_n)=\lambda_2 \quad\text{and}\quad \lim_{n\to\infty}[(u_{n+1}/u_n)-1]=0.$$

Hence

$$\lim_{n\to\infty}[-n\ln(u_{n+1}/u_n)]=\lim_{n\to\infty}[n\{1-(u_{n+1}/u_n)\}+\tfrac{1}{2}n\{1-(u_{n+1}/u_n)\}^2M(n)].$$

But, since λ_4 is finite, it follows that

$$\lim_{n\to\infty}[-n\ln(u_{n+1}/u_n)]=\lambda_4=\lim_{n\to\infty}\frac{\ln(1/u_n)}{\ln(n)}.$$

Now in the case $\lambda_4=+\infty$, given $N>0$, there exists n_0 such that $n\{1-(u_{n+1}/u_n)\}>N$ for all $n>n_0$. Then $\ln\{1-(N/n)\}>\ln(u_{n+1})-\ln(u_n)$. Hence

$$(N/n)[1+\tfrac{1}{2}(N/n)M(n)]<\ln(1/u_{n+1})-\ln(1/u_n),$$

$$N<\frac{N}{n\ln\{1+(1/n)\}}<\frac{\ln(1/u_{n+1})-\ln(1/u_n)}{\ln(n+1)-\ln(n)}$$

for all $n>n_0$. Then

$$\lim_{n\to\infty}\frac{\ln(1/u_{n+1})-\ln(1/u_n)}{\ln(n+1)-\ln(n)}=+\infty=\lim_{n\to\infty}\frac{\ln(1/u_n)}{\ln(n)}.$$

In the case $\lambda_4=-\infty$, the proof is similar.

Remark. The converse of the theorem is not true. To see this, let us consider the sequence $\frac{1}{2}, \frac{1}{3}, \frac{1}{2}, \frac{1}{3}, \cdots$. Then $\lim_{n\to\infty}[n\{1-(u_{n+1}/u_n)\}]$ does not exist, but $\lim_{n\to\infty}[\ln(1/u_n)/\ln(n)]=0$. Hence the relationship between λ_4 and λ_3 is analogous to that between λ_2 and λ_1.

References

1. E. Goursat, Cours d'Analyse Mathématique, Paris, 1943.
2. L. Brand, Advanced Calculus, New York, 1958.
3. T. J. I'A. Bromwich, An Introduction to the Theory of Infinite Series, London, 1949.

A METHOD FOR THE CONSTRUCTION OF SINGLY EVEN MAGIC SQUARES

JEROME ROTHSTEIN, Edgerton, Germeshausen and Grier Inc., Boston, Mass.

Books on mathematical recreations are replete with rules for constructing magic squares* of odd order, *i.e.*, $n=2m+1$. Doubly even magic squares, *i.e.*, those for which $n=4(2m+1)$ are easily constructed in several ways, *e.g.*, by "factoring" into $(2m+1)^2$ four by four blocks, or into sixteen $(2m+1)\times(2m+1)$ blocks. In the first case, the 4×4 magic square of Figure 1, with each number increased by $16(k-1)$ goes in place of k in any $(2m+1)\times(2m+1)$ magic square.

1	15	14	4
11	5	6	8
7	9	10	12
13	3	2	16

FIG. 1

The second construction puts a $(2m+1)\times(2m+1)$ magic square, with each number increased by $(2m+1)^2(k-1)$ in place of k in the 4×4 square. These methods fail in the case of the singly even square, $n=2(2m+1)$, because no magic square exists for $n=2$. This paper shows how to vary the pattern in 2×2 blocks systematically to yield a magic square of order $2(2m+1)$. (We have not discovered an analogous way to alter a block of four odd squares to yield a singly even one.)

The procedure is as follows: (1) Take

1	2
4	3

* By a magic square of order n we mean a square array of positive integers from 1 to n^2 inclusive such that the n rows, n columns, and two diagonals have the same sum. The sum can be shown to be $n(n^2+1)/2$. For an introduction to the field, see, for example, M. Kraitchik, Mathematical Recreations, Chap. VII, New York, 1942.

as a "standard" 2×2 block, and in place of the number k in a $(2m+1)\times(2m+1)$ magic square, put a standard 2×2 block with each number increased by $4(k-1)$. The resulting array will clearly have all its columns with the same sum, and this property is not destroyed if, within a 2×2 block, the upper and lower numbers of a column are interchanged (*e.g.*, 1 & 4 or 2 & 3).

(2) Alternate rows will now have the same sum, and the even rows sum to a value $4(2m+1)$ greater than the odd row sum. Therefore, if $2(2m+1)$ is added to the odd rows while the same amount is subtracted from the even rows, by the expedient of interchanging some upper and lower members of the same column within the same 2×2 block, all the rows will then have the same sum (which will be the same as the column sum). As each "left" column interchange of a 2×2 block shifts 3, while a "right" column shifts 1, $(m+1)$ left column interchanges and $(m-1)$ right column interchanges shift $3(m+1)+m-1=2(2m+1)$, thus leaving only the diagonals to be taken care of.

(3) The diagonal sums before the interchanges of (2) above differ by $2(2m+1)$. A left column interchange in a diagonal 2×2 block raises or lowers a diagonal sum by 3, a right column interchange in a diagonal 2×2 block raises or lowers a diagonal sum by 1. Both interchanges in the same diagonal 2×2 block raises or lowers a diagonal sum by 2. So if, for each diagonal, both interchanges are made in $(m-1)$ 2×2 diagonal blocks, and a left column interchange only is made in one 2×2 diagonal block, then one diagonal can be reduced by $2m+1$ while the other is augmented by the same amount. This would equalize the diagonals and give them the same sum as the columns.

(4) The requirements of (2) and (3) can be carried out simultaneously in many ways, a simple one being as follows: In the 2×2 blocks in the same column as the central 2×2 block and in the 2×2 block immediately to the left of the central block perform left column interchanges only. In the $(m-1)$ columns of 2×2 blocks to the right of the central block perform both right and left interchanges. In the border column of 2×2 blocks on the extreme right perform left interchanges only. The resulting square is magic as (1), (2) and (3) are satisfied. In Figure 2(a), a 5×5 square is exhibited, with a 10×10 square derived therefrom by the above rule shown in Figure 2(b).

Similar rules can be derived for other choices of the standard 2×2 block than that of (1).

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

FIG. 2(a)

65	66	93	94	4	2	32	31	60	58
68	67	96	95	1	3	29	30	57	59
89	90	17	18	28	26	56	55	64	62
92	91	20	19	25	27	53	54	61	63
13	14	24	22	49	50	80	79	88	86
16	15	21	23	52	51	77	78	85	87
37	38	45	46	76	74	84	83	12	10
40	39	48	47	73	75	81	82	9	11
41	42	69	70	99	98	8	7	36	34
44	43	72	71	97	100	5	6	33	35

FIG. 2(b)

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
JOHN R. MAYOR, AAAS and University of Maryland

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

RECENT RESEARCH IN MATHEMATICS: IMPLICATIONS FOR TEACHER EDUCATION*

PHILLIP S. JONES, University of Michigan

I shall interpret this title quite broadly, as I believe was intended. Research in pure and applied mathematics has long-range implications for teacher education. However, rather immediate and pressing implications follow from recent developments in the uses of mathematics and in experimentation and research in mathematics education. Since it is these implications which are our chief concern, and since that which I say here is intended to be the basis for the even more important discussion which is to follow, I shall turn first to a concise presentation of the current situation and from this deduce a list of implications which can serve as a basis for the following discussion out of which I hope constructive action will develop.

The major features of the current mathematics and mathematics-education landscape are as follows:

1. Mathematics, itself, both pure and applied, is growing at an unprecedented rate.
2. Both the range and amount of the applications of mathematics are ex-

* Presented at the Twelfth Annual Meeting of the American Association of Colleges for Teacher Education, February 11, 1960, Chicago, Illinois.

panding at perhaps an even faster rate and at a wide range of levels of difficulty and sophistication.

3. These two facts have led to an urgent need that more mathematics be understood earlier by those being trained in even the social sciences as well as in science, engineering, and technical vocational programs. In my mind, there is even a corresponding urgency for more mathematical background for all persons as a part of their general education. This latter belief is, however, more founded on philosophical ideas than on documented developments in research and application.
4. An unprecedented amount of study, discussion, experimentation and recommendations for reform in the content, and sequence of the curriculum and in teaching methods is being carried on, using large sums of money, and steadily releasing to teachers and administrators brochures, bibliographies, reports, and experimental classroom materials.

A brief explanation and documentation of these salient features of the mathematics-education landscape may help us to see both them and their implications more clearly. As I go along, I will point out that none of these four features is a post-sputnik eruption, although they have been played up, perhaps unfortunately, by the sputnik hysteria.

There is, of course, no easy way to measure the growth of mathematics. One measure, however, is the growth of its technical and research literature. Today's major mathematical abstracting journal, *Mathematical Reviews*, was founded in 1940 when World War II cut us off from European abstracting services. The 1945 issue contained 334 pages; 1950, 870; 1955, 1338. The number of new journals founded in the last twenty years includes *The Pacific Journal of Mathematics*, *Proceedings* and *Notices of the American Mathematical Society*, *The Arithmetic Teacher*, *Communications* and *Journal of the Association for Computing Machinery*, *Computers and Automation*, *Industrial Quality Control*, *Journal of Mathematics and Mechanics*, *Mathematics Student Journal*, *Journal of the Operations Research Society of America*, *SIAM Newsletter*, *SIAM Review*. In addition to these, several high quality local journals with international circulation have been founded by individual universities such as Illinois and Michigan, but even with an enforced condensation of articles, the flood of productive new ideas still produces a substantial publications backlog.

The titles of some of these new journals provide partial documentation of my second point; namely, that the range and amount of applications of mathematics have also been expanding at a terrific rate. Twenty years ago, and less, what are now substantial areas of application or even university departments were essentially unheard of or largely undeveloped: operations analysis, decision theory, information theory, linear programming, quality control, digital and analogue computation are illustrations. For example, in my school there is now a Professor of Mathematical Biology, who also bears the title "Senior Research Mathematician," located in our Mental Health Institute, a Professor

of Operations Analysis in the College of Engineering. There is an undergraduate course in Quantitative Methods in Biology, there are two courses in psychology which have calculus as a prerequisite, and there are several courses in mathematical economics. Notice that whereas in the past what little mathematics there was in the social sciences was largely statistics, this is no longer true. However, the growth of statistics itself has spurted and the subject has become much more mathematical. All of our engineers now get a little dose of probability and statistics in their freshman-sophomore years, and we now find it difficult to staff the several sections of a year's course in statistics for undergraduate engineers which course did not even exist a few years ago. The Aeronautical Engineering Department now requires all of its students to study theory of functions of a complex variable, and the earlier the better—a course which a few years ago was taken only by mathematicians and was usually a part of the graduate program.

I will leave for the discussion or for your later reading the further elaboration of this point and the fact that the roots of all of this run back before sputnik, and are, I believe, being nourished by the demands of a continuing and irreversible development of several aspects of our culture. I believe that in this discussion I have also documented my third point that there is an urgent need for *more* and *better selected* mathematics to be *understood* by *a much larger segment* of our population. This does not mean merely that we need more mathematicians, but that for both vocational and cultural reasons more persons must be mathematically *literate* and *appreciative* and at higher levels of literacy.

The fourth salient feature which I listed was the amount of study, discussion, reports, and materials relating to the mathematics curriculum and teaching methods which are bombarding teachers and administrators today. I will cite only one basis for this assertion. This little pamphlet, *Studies in Mathematics Education, A Brief Survey of Improvement Programs for School Mathematics*, was published in 1959 by Scott Foresman and Company. It listed only major and extensive programs. It is already out of date, but it listed four programs at the elementary level, thirteen at the secondary level, three at the college level, five under the heading "School Mathematics (Grades K-College)," and one under the heading of "Teacher Training."

All of this is fascinating and important, but we must march on to the major question "What are the implications of this for teacher education?" They are many, but they can be summarized in three statements:

1. Future teachers and administrators must be led to develop the motivation which will cause them throughout their professional careers to be alert to and informed about new programs, and regularly to go farther than this, even to the extent of carrying on continuing curricular and methodological changes and experimentation.
2. Future teachers and administrators must be helped to acquire the fundamental knowledge and understanding and background in both mathe-

matics and pedagogy which is essential to being able to read and critically evaluate the materials and proposals which will rain upon them.

3. These same motivations, knowledge and understanding must somehow be communicated to our in-service teachers and administrators.
4. Experimentation, research, evaluation must be continued in varied places and with varied approaches whether or not the major subsidized programs themselves continue or are replaced by others, or all die out.

These four statements may all seem to be truisms with nothing new in them. However, let us list quickly that which they seem to imply if we consider them in conjunction with the salient features of the present mathematics-education landscape which I listed earlier.

1. *The collegiate mathematics taught to undergraduates preparing to teach must be surveyed and compared with that which is needed to read, understand, and evaluate current proposals.*

This does not necessarily mean that a totally new mathematics program must be designed for those prepararing to teach. It may mean that the mathematics program as a whole for all college students needs revision or that a few special mathematics courses should be designed for and required of prospective teachers. There are three books that illustrate a rather extreme example of what I mean. The first is the experimental text dealing with matrices prepared for the second semester of the twelfth grade by a School Mathematics Study Group writing team last summer. The second is the report of the Commission on Mathematics of the College Entrance Examination Board printed in 1959 which recommended for this semester a choice of abstract algebra, probability and statistics, or introductory ideas of functions and calculus. The third is Volume I of a text, *Modern Mathematical Methods and Models*, which was prepared by the Committee on the Undergraduate Program of the Mathematical Association of America for a college sophomore course for persons with a major interest in the social sciences. It also treats matrices in an elementary way. A college using the CUP program would also have been teaching materials which would have prepared teachers to accept and teach, or to reject, the Commission's and the S.M.S.G.'s proposal. One can not intelligently either accept or reject that of which he knows nothing.

This example of matrices is rather extreme, and I am not sure that I myself favor their introduction into either the high school or as a required element in a teacher training program. However, teachers must somewhere learn more of such things as the foundations of the number system and of mathematics as a whole, of axiomatics, of set theory, of inequalities, of modern definitions of functions and relations, of probability and statistics.

2. *Methods courses in the teaching of mathematics must be given adequate time and be taught by persons who are both informed as to current trends, projects, and literature, and who also have an adequate knowledge of mathematics*

themselves. Mathematics teachers can not be well prepared by individual readings associated with a general methods course.

3. *Undergraduate and graduate programs must be planned with each other in mind.* That is, the undergraduate student should be carried far enough so that he may continue with some graduate study in mathematics. At the same time, graduate programs should begin at a level such that a well-prepared teacher may continue with some mathematics at the graduate level, and graduate offerings should always be planned to offer a varied sequence of courses appropriate for the secondary school teacher.

It is impossible to present all the desirable mathematics in a normal undergraduate teacher's certificate program. Substantial amounts of mathematics should be included in the graduate study program of all mathematics-educators. The development of advanced placement programs, seminars, enriched and accelerated programs from the seventh grade (and even lower) on up says that adequate background has never before been so important and that the junior high teacher must be included when we discuss these matters.

4. *Since many of the current programs lay great stress on the use of pupil discovery and heuristic teaching processes as the means to several ends such as understanding and appreciation of the nature and aliveness of the subject, devices must be found for communicating to students an understanding of and some practice in such techniques.* Care in the selection and training of supervising teachers, increased observation of well-taught classes, and practice in these processes under good close supervision will help. There are some movies in preparation which may help, and a clever and inspiring methods instructor may help teachers actually to write and "rehearse" materials and detailed plans for developmental approaches to new units.
5. *Extensive in-service training projects need to be formulated and staffed.* All of the recommendations made previously also apply to post-graduate and in-service training, but for this type of training to function most effectively special planning and added personnel are needed to meet the requests for extension courses, Saturday classes, continuing consulting aid, and in-service institutes. The college can often supply the initial push, the guidance and the secretarial help needed to initiate and maintain mathematics teachers' study groups and clubs.
6. Since much data is still needed and since nothing inspires both confidence and emulation in teachers more than observing or hearing of first-hand classroom experimentation, *education schools and staffs have a double responsibility to associate themselves with and to share in thoughtfully designed experimentation in mathematics education and in the psychological bases of mathematical learning, problem solving, and creativity.*
7. The roots of most mathematical ideas as well as the development of attitudes toward mathematics are to be found in the elementary schools. *Elementary school teachers must be trained both to understand and to enjoy*

elementary mathematics. In view of the total elementary teacher training program and the tendency of such teachers to be non-mathematical in their interests and background, it is probable that we need special mathematics courses for elementary teachers taught by sympathetic but mathematically sound and well informed persons.

In recent days we have heard much of "modern mathematics" and the lag of secondary school mathematics programs in comparison with the rapid advance of the subject itself. I do not believe that any topic should be deleted from the curriculum because of its age or added because of its modernity. However, I do believe that the curriculum must be the most intelligent selection possible from the totality of all that is important in mathematics today, whether old or new. To make such a selection one must first understand this mathematics and secondly select "intelligently." This latter means the selection must also take into account what types of training are important to the individuals and to the nation and also what of these materials can be taught effectively—meaningfully, for *understanding*—at various stages of pupil maturity. To do this teachers must understand how ideas develop within children and how the teacher functions to facilitate this development. There is much still to be learned of each of these, including how one passes along these insights and procedures to the coming generation of teachers.

West Virginia Study of the Education of Junior High School Mathematics Teachers

West Virginia University, in cooperation with the State Department of Education in West Virginia, is sponsoring an extended study of the preparation of junior high school mathematics teachers. This study has resulted in part from the recognition by the State Department of Education that the most critical shortage of teachers in West Virginia is in the area of junior high school mathematics, and also from the unique program in West Virginia for the certification of teachers for elementary and secondary schools by examination. One of the results of the project is hoped to be the preparation of an examination in mathematics to be used as a part of the West Virginia examination for certification of junior high school mathematics teachers.

The first step in the study was a two-day conference held at West Virginia University on December 7 and 8, 1959. Participants in the conference were leaders in some of the national mathematics curriculum studies, school administrators, and junior high school teachers in the state, as well as representatives of West Virginia University and the State Department of Education. The School Mathematics Study Group (SMSG) has granted permission to several junior high schools in West Virginia to try out the SMSG mathematics courses for the junior high school during the second semester this year and next year as a part of the study.

West Virginia University is sponsoring an in-service course for mathematics teachers during this year and is planning to continue this activity in 1960–61. In the summer of 1960, a study group will meet at West Virginia University to study the new mathematics curriculum materials and the ideas of modern mathematics needed by teachers in order to teach these materials, as a basis for the development of programs for the preparation of junior high school mathematics teachers, both preservice and in-service. The work started during the summer of 1960 will be continued during the academic year 1960–61 and, according to the plan, first steps will be taken in preparation of the examination to which reference was made earlier in this report.

Mathematics in a Science Fair

The Science Fair of Prince Georges County, Maryland for 1960 has chosen the theme "Mathematics." Each participant in the Science Fair is required to produce an exhibit in any field of science which he chooses but which will show the use of mathematics in this field of science. Quite a number of those interested in Science Fairs have been concerned about the lack of attention given to mathematics in these fairs. Prince Georges County appears to have found a unique solution to this problem for one year. For further information on plans in Prince Georges County, an inquiry might be addressed to Mr. Howard Owens, Science Supervisor, Prince Georges County Public Schools, Upper Marlboro, Maryland.

Conference on Elementary Science and Mathematics

The Oklahoma Frontiers of Science, Incorporated, sponsored a Conference on Elementary Science and Mathematics on December 5, 1959. The Conference was intended for school administrators not only in Oklahoma but in all parts of the United States. Sponsoring agencies for the Conference, in addition to the Oklahoma Frontiers of Science, were the American Association for the Advancement of Science, the American Association of School Administrators, and the Council of Chief State School Officers.

A paper on mathematics in the elementary school was presented by Dr. Paul Rosenbloom of the University of Minnesota. Copies of the report of the Conference may be obtained by writing to Dean James Harlow, College of Education, University of Oklahoma, Norman, Oklahoma.

Interest in Elementary School Mathematics Increases

The widespread interest in the improvement of the mathematics education in secondary schools is already bringing about similar interest at the elementary school level. During the past year a number of experimental studies at the elementary school level have been reported in this department. These include reports from the University of Illinois (May, 1959), Stanford University (June–July, 1959), Syracuse University (February, 1960).

Of major importance is the extension of the work of the School Mathematics Study Group to grades 4, 5, and 6. A panel on elementary school mathematics has been appointed. The members of the panel are: E. G. Begle, E. Glenadine Gibb, William T. Guy, Stanley Jackson, Irene Sauble, Marshall Stone, and J. Fred Weaver. A larger group met in Chicago in March for ten days to outline materials which will be written by a still larger group working at Stanford University in the summer. The materials to be prepared this summer probably will consist of a sample textbook for grade 4 and experimental units for grades 5 and 6. There will be teachers' commentaries for all materials prepared.

A conference sponsored by the National Council of Teachers of Mathematics in October has so far largely resulted in a decision by the Board of Directors of the Council to establish advisory committees on elementary and on secondary school mathematics which will formulate long-term plans for the improvement of mathematics education at these levels. The October conference was called to consider the overall future role of the National Council.

Among other recently organized curriculum studies in mathematics, not reported earlier in this Department, are the Washington Academy of Sciences sponsored program and another centered at the University of California, Berkeley, under the direction of Robert Karplus of the Department of Physics. Both of these studies are concerned with the integration of the teaching of mathematics and science, in the first instance at the junior high school level, and in the second, at the elementary school level. With foundation support, Catherine Stern of the New York City Public Schools and Anita Riess of

the University of Bridgeport are continuing some of their earlier work on the teaching of elementary mathematics, which involves in both instances the use of concrete developmental materials.

In the Washington area, the experimental seventh grade mathematics course of the University of Maryland is being tried out with gifted sixth grade pupils; the Washington, D. C., public schools have prepared a new course of study in arithmetic which has been considerably influenced by the courses at the junior high school level of SMSG and the University of Maryland. A parents' committee in Montgomery County, Maryland has requested the school administration to move quickly to the use of special teachers, with adequate mathematics preparation, in all classes in mathematics in the intermediate grades.

The AAAS-sponsored study on the use of special teachers of science and mathematics in grades 5 and 6 has been reported in this Department (October, 1959). A somewhat similar study with support from the Ford Foundation is being carried out in Ossining, New York, and Long Beach, L.I. Representatives of these investigations and other school systems using special mathematics and science teachers in the elementary school met in Washington in May.

In the summer of 1959 the National Science Foundation sponsored two summer institutes for supervisors and teachers of mathematics at the elementary school level. These were held at the University of Michigan and Rutgers University. A report on the Michigan Institute appeared in this Department (vol. 67, pp. 465–466). In 1960 there will be five institutes for teachers at this level. The 1960 institutes are located at Longwood College, the University of Michigan, the College of Saint Catherine, San Jose State College, and the University of Texas. In both summers a number of other institutes for elementary school administrators and teachers were concerned with teacher preparation in both mathematics and science.

AIBS Secondary School Biological Sciences Film Series

The American Institute of Biological Sciences is currently sponsoring a complete secondary school biology course which will make integrated use of more than 120 half-hour film lecture-demonstrations, classroom teaching and printed materials for students and teachers. A unique collaboration of nearly 100 of the nation's leading biologists is responsible for the preparation of the course content. In addition, nearly 100 secondary school teachers are serving as reviewers of the scripts and text materials. Funds to support preparation of the project have been made available by the Ford Foundation. The producer-distributor of the course materials is McGraw-Hill Book Company.

The course materials designed for the 10th grade are expected to be ready for distribution in the fall of 1960. The films will be available in color and black and white, and will be available in any combination desired—single films, topic units, or the entire series. Other materials which will be available are student study guides, teachers' manuals, discussion questions and suggestions for field and laboratory work, examinations, etc.

The Director of the project and continuing teacher in the films is Dr. H. Burr Roney, professor of biology at the University of Houston. Dr. Roney has had perhaps more experience in instructional television and films than any other teacher. The primary aim of the project is to provide maximum aid to the greatest number of secondary school biology teachers. This project is not considered an educational "cure-all" nor is it considered a replacement for the "live" classroom teacher.

More detailed information about the project may be obtained by writing to Mr. Jack Steuerwald, Assistant to the Director, Biological Sciences Film Series, American Institute of Biological Sciences, 2000 P Street, N. W., Washington 6, D. C.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1421. *Proposed by* (1) *S. P. Franklin and G. A. Hutchison, University of California at Los Angeles, and* (2) *W. R. Becker, New York City*

(1) Prove, for any integer $n>1$, $n/(n-1)>\{n^2/(n^2-1)\}^n$.

(2) Prove, for all $a>0$ and $b>0$, $\{(a+1)/(b+1)\}^{b+1}\geqq(a/b)^b$.

E 1422. *Proposed by R. W. Kilmoyer, Jr., Lebanon Valley College*

Find polynomials $f(x)$ such that $f(x^2)+f(x)f(x+1)=0$.

E 1423. *Proposed by Aboulghassem Zirakzadeh, University of Colorado*

Consider a plane closed curve possessing only ordinary points and a finite number of double points. Assign a positive direction to the curve and, starting from an ordinary point A, trace the curve in the given positive direction. Assign number 1 to the first double point met, 2 to the second double point met, and so on until you come back to A. Prove that of the two integers assigned to any double point, one is always even and the other always odd.

E 1424. *Proposed by V. E. Hoggatt and Charles King, San Jose State College*

A set of positive integers W_i, $i=1, 2, \cdots$, is *complete* if for each positive integer N there exists a subset W_{i_j}, $j=1, \cdots, k$, of set W_i such that $N=\sum_{j=1}^{k} W_{i_j}$.

(1) Show that the set of Fibonacci numbers $F_i(F_{i+2}=F_{i+1}+F_i,\ F_1=F_2=1)$ is complete.

(2) Show that if any one Fibonacci number is deleted from the set F_i, the set is still complete, but the deletion of two Fibonacci numbers renders the set incomplete.

E 1425. *Proposed by D. J. Newman, Brown University*

If a square lies within a triangle, prove that the area of the square does not exceed half the area of the triangle.

SOLUTIONS

An Application of the Chinese Remainder Theorem

E 1391 [1959, 915]. *Proposed by N. S. Mendelsohn, University of Manitoba*

Let $m_0, m_1, \cdots, m_r$ be positive integers which are pairwise relatively prime. Show that there exist $r+1$ consecutive integers $s, s+1, \cdots, s+r$ such that m_i divides $s+i$ for $i=0, 1, \cdots, r$.

Solution by J. E. Vinson, University of British Columbia. The problem is equivalent to showing the existence of a solution of the system of congruences

$$s \equiv -i \pmod{m_i}, \qquad i = 0, 1, \cdots, r.$$

Since the m_i are pairwise relatively prime it follows from the Chinese remainder theorem that solutions exist.

Also solved by R. G. Albert, R. G. Archibald, Ray Authement, D. W. Bailey, J. W. Baldwin, Leon Bankoff, H. F. Bechtell, W. J. Blundon, Brother Joseph Heisler, R. G. Buschman, H. R. Coomes, Underwood Dudley, E. S. Eby, George Glauberman, L. D. Goldstone, J. R. Guard, Virginia S. Hanly, J. H. Hodges, J. Hooley, Anne L. Johannsen, J. P. Jordan, Irving Katz, Donald Knuth, P. S. Landweber, H. R. Leifer, T. G. McLaughlin, Wallace Manheimer, D. C. B. Marsh, Sam Matlin, D. S. Passman, Walter Penney, Nathaniel Queen, R. T. Sandberg, C. M. Sandwick, Sr., Donna J. Seaman, Lawrence Shepley, G. J. Simmons, Charles Wexler, and the proposer. Late solutions by Alan Beal, Bill Emerson, Stanley Franklin and Gerald Hutchinson (jointly), D. P. Giesy, M. S. Klamkin, R. Q. Petersen, and C. F. Pinzka.

Queen established the stronger result that for any integers $d_0, d_1, \cdots, d_r$ there exists a number s such that m_i divides $s+d_i$ for $i=0, 1, \cdots, r$. As the proposer pointed out, the given problem solves, as a particular case, a recent Putnam Competition Problem in which it was required to show the existence of 1,000,000 consecutive positive integers each of which has a square factor.

Postage Stamp Dispensers

E 1392 [1959, 915]. *Proposed by R. K. Guy, University of Malaya, Singapore, Malaya*

I am confronted by stamp machines dispensing supplies of 3 cent and 4 cent stamps. I notice that if I have a sufficient supply of coins I can obtain stamps to the value of any whole number of cents except for 1 cent, 2 cents, and 5 cents. If the machines dispense only stamps of values m cents and n cents, how many exact values in cents can I not make up?

Solution by R. Z. Norman, Dartmouth College. Clearly there are infinitely many such unless we assume $(m, n)=1$. Under this assumption we use the following

LEMMA. *$mx+ny=c$ is solvable in nonnegative integers if and only if its complementary equation $mx+ny=mn-m-n-c$ is not.*

Proof. (1) We note that $mx+ny=c$ has exactly one integral solution (x_0, y_0) (its *principal solution*) for which $0 \leqq x_0 \leqq n-1$, and the solution is nonnegative if and only if $y_0 \geqq 0$.

(2) $mx+ny=mn-m-n$ has $(n-1, -1)$ as its principal solution.

(3) If (x_0, y_0) is the principal solution of $mx+ny=c$, then from (2) we see that the principal solution of its complementary equation is $(n-1-x_0, -1-y_0)$. Exactly one of these solutions is nonnegative.

Now for $c<0$ the equation never has a nonnegative solution, hence for $c>mn-m-n$ it always does. And it fails to have a nonnegative solution for exactly half the integers in $[0, mn-m-n]$, of which there are $(mn-m-n+1)/2 = (m-1)(n-1)/2$.

Also solved by R. G. Buschman, E. S. Eby, John Garnett and Brother T. Wesselkamper

(jointly), Michael Goldberg, L. D. Goldstone, J. P. Jordan, Donald Knuth, Sidney Kravitz, H. R. Leifer, Wallace Manheimer, D. C. B. Marsh, Sam Matlin, R. W. Means, J. K. Moore, D. S. Passman, Walter Penney, C. M. Sandwick, Sr., G. L. Simmons, and the proposer. Late solutions by Victor Baras, Alan Beal, and Bill Emerson.

Two Allied Determinants

E 1393 [1959, 915]. *Proposed by D. S. Mitrinovitch, University of Belgrade, Belgrade, Yugoslavia*

Evaluate the nth order determinant $D=|a_{ik}|$ for the two cases: (1) $a_{ik}=0$ when $i+k$ is even, (2) $a_{ik}=0$ when $i+k$ is odd.

Solution by R. A. Leonard, University of Cincinnati. Let r run through the even integers and s through the odd integers in the sequence $1, 2, \cdots, n$. (1) Using the Laplace expansion on the even rows of D we observe that each term in the expansion has as a factor a determinant with a column of zeros if n is odd and there is only one possibly nonzero term if n is even. Hence

$$D = (-1)^{n(n+1)/2} |a_{rs}| \, |a_{sr}|, \ n \text{ even}; \qquad D = 0, \ n \text{ odd}.$$

(2) In the Laplace expansion on the odd rows of D all terms but one vanish identically and, for $n>1$, $D=|a_{rr}| \, |a_{ss}|$. If $n=1$, then $D=a_{11}$.

Also solved by R. G. Albert, H. F. Bechtell, J. M. Chambers, J. P. Jordan, Donald Knuth, P. S. Landweber, Kenneth Loewen, R. C. Luebbe, D. C. B. Marsh, Walter Penney, Thomas Porsching, Donna J. Seaman, and T. L. Vanden Eynden. Late solutions by D. W. Bailey and Anna and R. G. Thompson (jointly).

A Minimum Tetrahedral Volume

E 1394 [1959, 916]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given a trihedral angle and a point within it, construct the plane through the point which intercepts on the trihedral angle the tetrahedron of minimum volume.

Solution by L. D. Goldstone, New York State Public Works Lab., Albany, N. Y. Take oblique cartesian axes in 3-space along the edges of the given trihedral angle and let the given point P have coordinates (a, b, c). Let u, v, w be the intercepts on the x, y, z axes of a plane through P; then $a/u+b/v+c/w=1$. Designate the angle between the y and z axes by θ, and the angle between the x axis and the yz plane by ϕ. It is then an elementary exercise to show that the volume of the tetrahedron cut from the given trihedral angle by the plane through P is $V=kuvw$, where $k=\sin\theta \sin\phi$. Now, by the arithmetic-geometric mean theorem we have

$$(a/u + b/v + c/w)/3 = 1/3 \geqq (abc/uvw)^{1/3},$$

or $uvw \geqq 27abc$. We now see that V will be a minimum when $uvw=27abc$, and this can occur only when $a/u=b/v=c/w=1/3$. We now conclude that the plane through P cutting off the tetrahedron of minimum volume intersects the tri-

hedral angle in a triangle having P for centroid and is easily constructed from its intercepts $u=3a$, $v=3b$, $w=3c$.

Also solved by Brother T. Wesselkamper, Michael Goldberg, C. B. Grosch, Donald Knuth, D. C. B. Marsh, and the proposer. Late solution by M. S. Klamkin.

Editorial Note. A more difficult, and still unsolved, problem is that of determining the plane through P which intersects the given trihedral angle in a triangle of minimum area. The plane analog of this problem is the famous "Philo's line" problem, and is beyond the euclidean tools. The plane analog of E 1394 is easy.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4911. *Proposed by Peter Ungar, New York University*

Let $f(z)=\lambda z+a_2z^2+\cdots$ be an entire function where $|\lambda|>1$. Let N be any neighborhood of the origin. The images of N under successive iterations of f cover the whole plane with the possible exception of one point. The same is true if $\lambda=1$ and $f(z)\not\equiv z$.

4912. *Proposed by Paul Erdös, University of Toronto*

Prove: there exists an absolute constant $c_1>0$ such that, for every n, there always exist integers a_i $(i=1, 2, 3)$, $n<a_i<n+n^{1/2}+c_1n^{1/4}$, for which a_1 is a divisor of a_2a_3. If $c_2>0$ is sufficiently small this is false; in fact, there are infinitely many values of n such that, for every triplet, $n<a_i<n+n^{1/2}+c_2n^{1/4}$ we must have that a_1 does not divide a_2a_3.

4913. *Proposed by Paul Erdös, University of Toronto*

Let w be any positive integer and let a_i, a_j be integers satisfying $w^2\leqq a_i, a_j<(w+1)^2$. Prove that all products a_ia_j are distinct.

4914. *Proposed by Leo Moser, University of Alberta*

For any positive integers m, n, prove that K is an integer, where

$$K=\frac{(mn)!1!2!\cdots(m-1)!1!2!\cdots(n-1)!}{1!2!\cdots(m+n-1)!}.$$

4915. *Proposed by S. W. Golomb, California Institute of Technology*

Show that, if the trinomial $f(x)=x^n+x^a+1$ over GF(2), with $0<a<n$, has any repeated factors, then $f(x)$ is a perfect square.

4916. *Proposed by Olga Taussky, California Institute of Technology*

Fuglede's theorem (*Proc. Nat. Acad. Sci.* 36 (1950), 35–40) specialized to finite matrices over the complex numbers states that, for A normal, the relation $AB=BA$ implies $A^*B=BA^*$. Putnam's generalization (*Amer. J. Math.* 73 (1951), 357–362) states that if A_1, A_2 are normal and if $A_1B=BA_2$ then $A_1^*B=BA_2^*$.

Find, conversely, a necessary and sufficient condition on a nonsingular n by n matrix B such that the relations $A_1B=BA_2$ and $A_1^*B=BA_2^*$ imply that A_1 and A_2 are normal.

SOLUTIONS

Entire Function

4857 [1959, 595]. *Proposed by I. N. Baker, University of Alberta, and D. J. Newman, Brown University (independently)*

What is the most general entire function which takes real values on the real axis only?

Solution by D. J. Newman. Let $f=u+iv$. It follows from the hypothesis that v is of one sign in the upper half plane, say it is positive. When $v=0$ (on the real axis) we conclude then that $\partial v/\partial y\geqq 0$, whence $\partial u/\partial x\geqq 0$. Hence f takes each real value at most once. By Picard's theorem f must be linear.

Also solved by I. N. Baker, J. H. Reill, and M. S. Robertson.

Editorial Note. Robertson generalizes the problem as follows: *Let $f(z)$ be an entire function, real on the real axis, for which $I\,f(z)$ changes sign exactly $2p$ times on every circle $|z|=R\geqq R_0>0$. Then $f(z)$ is a polynomial of degree $\leqq p$ with real coefficients.*

Fredholm Integral Equation

4859 [1959, 595]. *Proposed by Richard Bellman, The RAND Corporation*

Consider the Fredholm integral equation

$$u(x) = v(x) + \int_a^1 k(x, y)u(y)dy, \qquad a < 1,$$

with the solution

$$u(x) = v(x) + \int_a^1 K(x, y, a)v(y)dy.$$

Under appropriate conditions on $k(x, y)$ show that the resolvent kernel satisfies the equation $\partial K(x, y, a)/\partial a = -K(x, a, a)K(a, y, a)$.

Solution by I. I. Kolodner, University of New Mexico. Assume that the resolvent K and its partial derivative K_a exist. As is well known, K is the unique solution of the following Fredholm equations:

$$K(x, y, a) = k(x, y) + \int_a^1 k(x, z)K(z, y, a)dz, \tag{1}$$

$$K(x, y, a) = k(x, y) + \int_a^1 K(x, z, a)k(z, y)dz. \tag{2}$$

From (1) one gets, on differentiating with respect to a,

$$K_a(x, y, a) = -k(x, a)K(a, y, a) + \int_a^1 k(x, z)K_a(z, y, a)dz.$$

This shows that K_a is the solution of the same Fredholm equation with $v(x)$ replaced by $-k(x, a)K(a, y, a)$. Using the resolvent formula one concludes that

$$K_a(x, y, a) = -k(x, a)K(a, y, a) - \int_a^1 K(x, z, a)k(z, a)K(a, y, a)dz.$$

In view of (2), this gives

$$-\int_a^1 K(x, z, a)k(z, a)dz = k(x, a) - K(x, a, a),$$

and the desired result follows on making this substitution.

Also solved by J. L. Brown, Jr., Emil Grosswald, H. J. Zimmerberg, and the proposer.

Editorial Note. Bellman has already shown (*Proc. Amer. Math. Soc.*, vol. 8 (1957), pp. 435–440) that if $k(x, y)$ is symmetric, continuous in both variables on $0 \leqq x, y \leqq 1$, and

$$-\int_0^1 \int_0^1 k(x, y)u(x)u(y)dxdy + \int_0^1 u^2dx > 0,$$

then $K_a(x, y, a) = -K(a, x, a)K(a, y, a)$, where $0 \leqq a \leqq 1$.

Symmetric Functions of Roots

4860 [1959, 596]. *Proposed by Immanuel Marx, Purdue University*

Given the polynomial $F_n(x) = x^n - a_1x^{n-1} + a_2x^{n-2} - \cdots + (-1)^n a_n$, and integers p and q, $0 < q \leqq p \leqq n$. From each subset of the roots of F_n taken p at a time suppose formed the elementary symmetric functions of degree q, *i.e.*, the sum of all products of distinct members of the subset taken q at a time. Knowing only the coefficients of F_n, devise a method to construct a polynomial whose roots are the $\binom{n}{p}$ symmetric functions described. Illustrate the method by deriving from the quartic $F_4(x) = x^4 - a_1x^3 + a_2x^2 - a_3x + a_4$, having (unknown) roots r_1, r_2, r_3, r_4, the quartic whose roots are $R_1 = r_2r_3 + r_3r_4 + r_4r_2$, $R_2 = r_1r_3 + r_3r_4 + r_4r_1$, $R_3 = r_1r_2 + r_2r_4 + r_4r_1$, $R_4 = r_1r_2 + r_2r_3 + r_3r_1$; in other words, the case $n = 4$, $p = 3$, $q = 2$.

Solution by the proposer. Divide the polynomial F_n formally by the polynomial

$$D_p = x^p - b_1x^{p-1} + b_2x^{p-2} - \cdots + (-1)^p b_p,$$

not until the remainder is of degree $p-1$, but only until the remainder is of degree p; let this remainder be

$$R_p(x) = c_0x^p - c_1x^{p-1} + c_2x^{p-2} - \cdots + (-1)^p c_p.$$

If $D_p(x)$ is an exact factor of $F_n(x)$—and this means, if the roots of D_p are a subset of the roots of F_n—then $R_p(x) = c_0D_p(x)$, so that $c_1 - c_0b_1 = 0$, $c_2 - c_0b_2 = 0, \cdots, c_p - c_0b_p = 0$.

The functions $c_0, c_1, \cdots, c_p$ are polynomials (of degree at most $n-p$) in the quantities $b_1, b_2, \cdots, b_p$, with coefficients linear in $a_1, a_2, \cdots, a_n$. Elimination of all but b_q among the p relations $c_j - c_0b_j = 0$ gives a polynomial equation in b_q with coefficients that are polynomials in $a_1, \cdots, a_n$, and this equation must be satisfied whenever b_q is the appropriate symmetric function of a subset of p roots of F_n.

For $n=4$, $p=3$, $q=2$, one obtains in this way the polynomial

$$x^4 - 2a_2x^3 + (2a_4 + a_2^2 + a_1a_3)x^2 - (2a_2a_4 - a_1^2a_4 + a_3^2 + a_1a_2a_3)x + (a_4^2 + a_2a_3^2 - a_1a_3a_4).$$

Also solved by Chung-lie Wang, and C. C. Yalavigi.

Editorial Note. The proposer comments that the problem was suggested by Lin's iterative method for factoring polynomials (*Journ. Math. Phys.*, 20 (1941), p. 231). An explicit form for the coefficients c_j is indicated in a discussion of Lin's method by Aitken (*Proc. Roy. Soc. Edinburgh*, 63 (1951), p. 174).

Integral Transform

4861 [1959, 596]. *Proposed by Joseph Lehner, Michigan State University*

Find a function $f(x)$, nonnegative and bounded on the closed interval $[0, \infty]$, such that

$$\limsup_{t\to\infty} \left| \int_0^t f(x)e^{2\pi i\lambda x}dx \right| = +\infty$$

for all real λ with the exception of a countable set.

Solution by the proposer. Define

$$f(x) = \begin{cases} 1, & \nu^2 < x < \nu^2 + 1, \\ 0, & \text{elsewhere.} \end{cases} \qquad \nu = 0, 1, 2, \cdots,$$

Let $\phi(t, \lambda) = \left| \int_0^t f(x)e^{2\pi i\lambda x}dx \right|$. Then for $\lambda \neq 0$,

$$\phi(n^2, \lambda) = \left| \sum_{\nu=0}^{n-1} \int_{\nu^2}^{\nu^2+1} e^{2\pi i\lambda x}dx \right| = \left| \sum_{\nu=0}^{n-1} e^{2\pi i\nu^2\lambda} \right| \cdot \left| \frac{e^{2\pi i\lambda} - 1}{2\pi i\lambda} \right|.$$

Hardy and Littlewood (*Acta Math.*, 1914, p. 225) have shown that the sum in the right member exceeds $C\sqrt{n}$ as $n\to\infty$ on a certain sequence, when λ is irrational. Hence,

$$\limsup_{t\to\infty}\phi(t,\lambda)\geqq\limsup_{t\to\infty}\phi(n^2,\lambda)=+\infty,\ \lambda\text{ irrational.}$$

This is the desired result, but it is also easy to treat the case of rational λ by means of Gaussian sums. When λ is an integer $\neq 0$, $|\phi(t,\lambda)|\leqq 1$.

Distinct Numbers of Prescribed Form

4862 [1959, 596]. *Proposed by D. J. Newman, Brown University*

Given a set S of distinct numbers $a_1,\cdots,a_n$. Define $k=k(S)$ to be the number of numbers of the form a_i+2^j, $j=1,2,\cdots,n$. What is $\min_S k(S)$?

Solution by D. C. B. Marsh, Colorado School of Mines. Array the numbers, a_i+2^j, as shown:

$$\begin{aligned}&a_1+2,\ a_1+4,\cdots,a_1+2^n\\&a_2+2,\ a_2+4,\cdots,a_2+2^n\\&\cdots\cdots\cdots\cdots\\&a_n+2,\ a_n+4,\cdots,a_n+2^n.\end{aligned}$$

Obviously, no row may contain a repeated value. It is also easy to show that when the a_i are distinct, no pair of rows may contain more than one common number. Thus the maximum number of numbers which may be repeated in any such array will be ${}_nC_2$ at best, if it is possible for each distinct pair of rows to contain a different common element. In such a case, $\min_S k(S)=n^2-{}_nC_2={}_{n+1}C_2$. That this minimum may actually be attained is evidenced by the particular set: $a_i=a+2^i$, with a arbitrary.

Also solved by J. H. B. Kemperman, and the proposer.

Extending the Triangle Inequality

4863 [1959, 728]. *Proposed by W. F. Stinespring, University of Chicago*

For what constant k (if any) can the inequality

$$|A+B|\leqq k(|A|+|B|)$$

be established for all n by n real symmetric matrices ($n\geqq 2$)? Here the notation $|A|$ means the positive square root of A^2, and the relation $\leqq$ between two matrices means that their difference is nonpositive definite.

Solution by Chandler Davis and W. F. Stinespring. No such inequality is valid (even if k depends on n). Let

$$A=\begin{pmatrix}1 & t\\ t & t^2\end{pmatrix},\qquad B=\begin{pmatrix}-1 & t\\ t & -t^2\end{pmatrix}.$$

Because $A \geqq 0$, $|A| = A$; because $B \leqq 0$, $|B| = -B$. Thus $|A| + |B| = A - B$, a diagonal matrix with eigenvalues 2, $2t^2$. On the other hand, $A + B$ has eigenvalues $\pm 2t$, so $|A + B|$ is the constant matrix $2t$. For $|A + B| \leqq k(|A| + |B|)$ here requires $2t \leqq k \cdot 2$ and $2t \leqq k \cdot 2t^2$. No matter how large k is chosen, sufficiently small positive t violates the second inequality.

Remark. It is easy to show that, for any given A, B, there exists some k such that the inequality holds. Even this weaker statement fails, however, in the infinite-dimensional case.

Also solved by Ken Alles, and R. D. Gordon.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Matrix Calculus (2nd ed.) By E. Bodewig. Interscience, New York, 1959. xi+452 pp. $9.50.

This book is true to its title, and almost as current as a seminar with its comparisons of new methods and preoccupation with credit to the proper originators. Its four parts are titled Matrix Calculus, Linear Equations, Inversion of Matrices, and Eigenproblems. Direct and iteration methods are both given in detail with full references, and special cases are discussed. The notation is not completely standard, the author explaining in his preface that his new notation is intended to build on the fact that matrix arithmetic uses the rows and columns of matrices rather than their individual elements, for the most part. His point seems to me well taken and his notation adequate, but what might have been a service in a textbook introducing matrices from the vector space approach is likely to prove merely an annoyance in this compendium of methods, which is a reference work. Several slips in reference numbers got by the editors, and the English is not always idiomatic, though it is always clear enough. In any case, though the notation or possibly a wrong reference number may hold up the user for a few minutes, he will find it worth while to check through this book if he has a hard matrix to deal with.

MARGARET W. MAXFIELD
Naval Ordnance Test Station,
China Lake, California

Irreducible Tensorial Sets. By U. Fano and G. Racah. Academic Press, New York, 1959. vii+171 pp. $6.80.

A unified treatment of quantities (such as vectors, tensors, eigenstates) which experience a *linear* transformation when the system of space coordinates rotates is given for the benefit of physicists interested in the theory of coupling and recoupling of angular momenta. Though scattered examples justify the general title, the burden of the material deals with the quantum mechanics of rotation. The authors give two pages of bibliography from Casimir (1931) to Wigner (1958).

ARTHUR BERNHART
The University of Oklahoma

Analytic Function Theory, Vol. I. By Einar Hille. Ginn, Boston, 1959. xi+308 pp. $6.50.

It would be hard to imagine a better textbook in complex function theory for a senior or first year graduate course than the volume under review. The table of contents is essentially canonical: 1. Number systems; 2. The complex plane; 3. Fractions, powers, and roots; 4. Holomorphic functions; 5. Power series; 6. Some elementary functions; 7. Complex integration; 8. Representation theorems; 9. The calculus of residues; Appendix A. Some properties of point sets; Appendix B. Some properties of polygons; Appendix C. On the theory of integration. However, there are many interesting variations in the book itself. The symbolism of Boolean algebra is used where it will do the most good; the book is studded with historical footnotes which do much to emphasize the evolutionary character of mathematics; the author's own interest in abstract analysis is evidenced by his description of important function spaces and algebras as they appear in his exposition; important facts about complex functions of a real variable are given (often in exercises). The proofs are exact, and for the most part maximally simple. Definitions are precise. Some teachers may wish to rearrange the order somewhat, for example so that students will meet a variety of analytic functions early in the course (e^z is defined on page 138). Fortunately the book is so written that reasonable permutations cause little confusion. For students who already know some real variable theory beyond the usual advanced calculus, the book may be somewhat elementary. An enterprising teacher can still use it when teaching such students, however, omitting some items and supplying more theory *ad libitum* (*e.g.*, the general Jordan curve theorem, elliptic functions, Riemann surfaces, the great Picard theorem, etc.).

Professor Hille brings to the writing of this book a seldom-rivalled depth of scholarship in both the classical and abstract parts of his subject, together with a lucid and witty prose style. Accordingly his book must go into the select library of "What every young analyst should know."

EDWIN HEWITT
University of Washington

An Introduction to Differential Geometry. By T. J. Wilmore. Oxford University Press, New York, 1959. 314 pp. $5.60.

This book offers much more than its modest title would lead one to expect. Although it is not intended to be a comprehensive account of classical differential geometry, a sizeable amount of the metric theory of curves and surfaces in three-dimensional Euclidean space is packed into Part 1, all done very neatly by use of vector methods. Following three chapters on curves and intrinsic and extrinsic properties of surfaces, Part 1 is concluded by a fourth chapter on geometry of surfaces in the large which includes material heretofore not seen in an English text. For an understanding of this material on global geometry the reader needs some familiarity with such concepts as compactness, Hausdorff space, homeomorphism, and covering space (which are not defined or explained here). However, this chapter is independent of the remainder of the book and may be omitted. In the author's view the material of Part 1 should be covered before the reader encounters the more general tensor theory in Part 2.

Under the general heading of differential geometry in n-dimensional space in Part 2, there appears a chapter each on tensor algebra, tensor calculus, Riemannian geometry, and applications of tensor methods to surface theory. Vector spaces and the dual space are introduced and then follows the tensor product of vector spaces. Differentiable manifolds, connections, and generalized covariant differentiation constitute part of the chapter on tensor calculus. The following chapter on Riemannian geometry portrays the tensor analysis of a special object—that of a field of symmetric, second-order, covariant tensors. In the concluding chapter Riemannian geometry is specialized to a very brief account of the use of tensors in ordinary surface theory. In the progressive specialization of the latter part of the book, the author appears to reverse his earlier conviction on order of presentation.

The book is admirably written by one who is familiar with current research in the field. Mention is made of various unsolved problems. Useful references are given for further reading. Much material is covered in the rich lists of exercises. Most of the sixty-six exercises in the final list are taken from examination papers of the University of Liverpool. The book is highly recommended as a textbook for a two-semester sequence on the essential ideas and methods of differential geometry.

C. E. Springer
The University of Oklahoma

Introduction to Algebraic Geometry. By Serge Lang. Interscience, New York, 1958. xi+260 pp. $7.25.

This book is written not only as an introduction to algebraic geometry, but as an introduction to Weil's *Foundations of Algebraic Geometry.* The author says that the purpose of the book is to give a self-contained introduction to the algebraic aspects of the "algebraico-geometric" method of studying algebraic

geometry without presupposing any extensive knowledge of algebra. It fulfills its purpose very well.

The book begins with a discussion of places, valuations, and extensions of places. Then algebraic varieties (affine, projective, and abstract) are introduced. Here are included the Hilbert Nullstellensatz and classical elimination theory. There is essentially no intersection theory in the text. The third chapter, on absolute theory of varieties, contains much of the material in the first chapter of Weil's *Foundations*. There are chapters on products, projections, and correspondences; on normal varieties (included here is Zariski's original proof of his Main Theorem on birational correspondences); on divisors and linear systems; on differential forms; and on simple points. There is a short chapter on connected algebraic groups, which includes theorems of a general nature which have been used recently in laying the foundation of the theory of algebraic groups. The final chapter contains Weil's proof of the Riemann-Roch theorem for curves and a sketch of the construction of the Jacobian variety.

The book is written in a concise and readable style. There are no exercises, but the proofs of some theorems are left to the reader. However, there is plenty of encouragement for the student to delve further into the subject; at the end of nearly every chapter there are references to pertinent papers, many very recent.

ALICE T. SCHAFER
Connecticut College

Mathematics of Finance. By Edwin D. Mouzon, Jr. and Paul K. Rees. Allyn and Bacon, Boston, 1959. xii+464 pp. $7.95.

This is a well-written book designed for a three-semester-hour course for beginning students in the mathematics of finance. In addition to the topics usually covered in such a course the authors have provided a substantial amount (about thirty per cent of the text) of background material in percentage and in elementary algebra through geometric progressions, enough for an introductory course in the rudiments of algebra and business arithmetic. The topics included in the main part of the text are statistics (14 pages); simple and compound interest; simple and general annuities; amortization and sinking funds; depreciation and capitalized cost; bonds; life annuities and life insurance.

The authors have exercised great care in the formulation of definitions and in the exposition of the theory, illustrating them copiously with good examples worked out in detail. The stated aim of the authors to make this textbook suitable for self-study as well as for classroom use is, in the reviewer's estimation, fully achieved.

There are numerous well-graded exercises with answers supplied for about three-fourths of them. The book is also provided with ten tables, including a five-place table of logarithms, eight financial tables and a mortality table.

J. M. FELD
Queens College

Analysis of Linear Systems. By David K. Cheng. Addison-Wesley, Reading, Massachusetts, 1959. xiii+431 pp. $8.50.

Professor Cheng has written an excellent text for engineers who need to know the useful mathematical techniques for solving linear ordinary differential equations. He spends considerable effort on the translation from the physical problem to the mathematical problem. This translation is, of course, the principal source of error in the analysis of physical systems. He has many excellent examples and good sets of problems at the end of each chapter.

The student who has mastered this book will have a working knowledge of Fourier series, Laplace transforms and inverse transforms, and the mathematical tools for determining stability of linear systems, such as the Routh-Hurwitz criterion and the Nyquist diagram. The mathematician will have little to complain about with respect to rigor, for the author gives ample references for complete proof of the points he wishes to treat only briefly.

The reviewer recommends this as an excellent textbook for the electrical and mechanical engineer and certainly a desirable reference for the physicist or mathematician who is interested in applying his theory.

JOHN E. MAXFIELD
Naval Ordnance Test Station
China Lake, California

Analysis of Straight-Line Data. By Forman S. Acton. Wiley, New York, 1959. xiii+267 pp. $9.00.

This book has been written with enthusiasm, clarity and care. The author has a pleasant breeziness of description that helps to emphasize important ideas. Well-chosen examples are used to illustrate and explain the text.

Due to his conviction that more detailed expositions of classical and modern statistical techniques are needed the author has selected topics pertinent to the engineer or physical scientist interested in data containing one or more lines entrapped within his experimental variability. The author hopes that the experimentalist himself will be able to use the analytical techniques introduced. The following quotation is from the author's preface. "The structure of the experiment determines the proper method for analyzing its data, and the experimenter himself knows that structure best. Since his statistical background is apt to be shaky, he needs to be guided—at times in considerable detail—and he usually prefers concrete examples from which general principles may be inferred. I have therefore tried to introduce each subject with data from a real experiment—examining them at first broadly, and then again with an increase in detail that occasionally they do not deserve. When I have felt that they will not mislead I have sometimes relied on explanations that are heuristic, but in general I have tried to emphasize the essential questions that are posed by the mathematical models our experimenter may choose in analyzing his data. It is with the influence of the model on the nature of the extractable information that

I have been chiefly concerned." The reader of this work will be convinced that the author has succeeded most admirably in carrying out his intentions.

The text discusses clearly and carefully with examples and illustrations the following topics: the classical model in which x is known without error and the variance of y is constant, regression with several values of y for each known x, samples from bivariate normal populations, regression with both x and y in error, several lines and the analysis of variance, the exposure of curvature and the use of orthogonal polynomials, use of transformations, rejection of unwanted data, and cumulative data. In addition there is a good list of current references plus a bibliography which includes helpful comments. The appendix contains in 22 pages the necessary statistical tables.

JOHN C. BRIXEY
The University of Oklahoma

Probability and Related Topics in Physical Sciences. By Marc Kac with special lectures by G. E. Uhlenbeck, A. R. Hibbs, and B. van der Pol. Interscience, New York, 1959. xiii+266 pp. $5.60.

This monograph is Volume I in the series *Lectures in Applied Mathematics* and grew out of lectures delivered by the author at the Summer Seminar, Boulder, Colorado, 1957. To sum up, the book is a notable contribution in the general area of probability and statistical mechanics written in a fresh, appealing style and represents an auspicious beginning for the series. Kac succeeds altogether in capturing the reader's fancy with his choice of examples, frequently unfamiliar and ingenious, and his methods of proof, often not quite standard, often interlarded with burdensome detail, but always challenging.

In a sense, no review of this book is necessary since the author includes an excellent account of its content and purpose in his preface. It is well to point out additionally that, although the subjects treated seem to change almost kaleidoscopically at times, there is an underlying idea which gives the book a kind of unity, and this idea is, of course, measure theory. Indeed, perhaps the greatest weakness of the book is that the theory of measure as such is bypassed, and it is reasonable to suppose that a reader without this background could be partially overwhelmed in consequence. It is to be hoped that this omission will not make the book inaccessible to physicists who could certainly derive as much profit from it as mathematicians, but the going does get a bit rough in Chapter IV. (Integration in Function Spaces.) As a compensation, there is a brief article on quantum mechanics by A. R. Hibbs among the appendices which is quite elementary and charmingly written.

Kac has annotated his text with a number of footnotes, mostly historical, which are likewise most engaging. One wishes there were even more of them! In fact the only quarrel to be picked with the author is his statement on page 23 that the problem of constructing Hadamard matrices is still unsolved. This ignores the beautiful work done by Paley (1933) on this problem as well as more

recent contributions. Curiously, the partial solution by Paley has a distinctively Kacian flavor!

KENNETH A. BUSH
University of Idaho

BRIEF MENTION

Calculus with Analytic Geometry (2nd ed.). By Richard E. Johnson and Fred L. Kiokemeister. Allyn and Bacon, Boston, 1960. xiii+709 pp. $8.25.

The most notable change in the second edition of this excellent text is the inclusion of work on vectors and the addition of over 400 supplementary exercises. If you plan to give a course in analytic geometry and calculus, this book is certainly one which merits consideration as a possible text.

Measurement: Definitions and Theories. Edited by C. West Churchman and Philburn Ratoosh. Wiley, New York, 1959. viii+274 pp. $7.95.

A collection of authoritative essays derived from the A.A.A.S. symposium held in New York City in 1956.

The Scientific American Book of Mathematical Puzzles and Diversions. By Martin Gardner. Simon and Schuster, New York, 1959. xi+178 pp. $3.50.

The rich and varied contents of this volume under the skilled pen of Martin Gardner will delight the mathematician and nonmathematician, alike. It is too bad that the article on flexagons fails to recognize the excellent paper by Oakley and Wisner which appeared in the March 1957 issue of this MONTHLY.

An Introduction to Statistical Mechanics. By J. S. R. Chisholm and A. H. de Borde. Pergamon Press, New York, 1958. ix+160 pp. $6.00.

Designed for an undergraduate course in statistical mechanics, this book would be more suitable for American use if it contained a reasonable selection of problems and exercises for student consideration.

Magnets: The Education of a Physicist. By Francis Bitter. Doubleday, New York, 1959. 155 pp. 95¢.

Echoes of Bats and Men. By Donald R. Griffin. Doubleday, New York, 1959. 156 pp. 95¢.

Soap Bubbles and The Forces Which Mould Them. By C. V. Boys. Doubleday, New York, 1959. 156 pp. 95¢.

How Old is the Earth? By Patrick M. Hurley. Doubleday, New York, 1959. 160 pp. 95¢.

The Neutron Story. By Donald J. Hughes. Doubleday, New York, 1959. 158 pp. 95¢.

Mathematics Tests Available in the United States. By Sheldon S. Myers. National Council of Teachers of Mathematics, Washington D. C., 1959. 12 pp. 50¢.

The Supervisor of Mathematics, His Role in the Improvement of Mathematics Instruction. Secondary School Curriculum Committee, National Council of Teachers of Mathematics, Washington D. C., 1959. 10 pp. 15¢.

A Guide to the Use and Procurement of Teaching Aids for Mathematics. By E. J. Berger and D. A. Johnson. National Council of Teachers of Mathematics, Washington, D. C., 1959. iii+41 pp. 75¢.

Nomography, Second Edition. By Alexander S. Levens. Wiley, New York, 1959. viii +296 pp. $8.50.

An extensive revision and expansion of Levens' 1948 work on alignment charts and concurrency nomograms.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., *University of Buffalo*

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

University of Colorado: Assistant Professor L. W. Rutland has been promoted to Associate Professor in the Department of Applied Mathematics; Assistant Professor P. F. Hultquist has been promoted to Associate Professor in the Department of Applied Mathematics and appointed Executive Director of the Numerical Analysis Center.

Olivet Nazarene College: Assistant Professor B. F. Hobbs has been promoted to Acting Chairman of the Department of Mathematics; Mr. P. F. Tomlinson has been promoted to Assistant Professor.

Western Michigan University: Professor J. H. Powell has been appointed Head of the Mathematics Department, effective July 1, to succeed Professor Charles Butler. Professor Butler has asked to be relieved of administrative duties in order to return to full-time teaching.

Mr. R. L. Arms, University of Michigan, has accepted a position as Mathematician with the Bendix Aviation Corporation Computer Division, Los Angeles, California.

Mr. R. V. Borchers, U. S. Naval Proving Ground, Dahlgren, Virginia, has accepted a position as Mathematician with the National Aeronautics and Space Administration, Washington, D. C.

Mr. C. M. Bruen, I.B.M. Scientific Computation Laboratory, Endicott, New York, has accepted a position as Research Specialist with the Aerospace Division of the Boeing Airplane Company, Seattle, Washington.

Mr. N. D. Cohen, Los Angeles State College, has accepted a position as Mathematician with the Burroughs Corporation, Pasadena, California.

Mr. C. E. Diesen, Bell Aircraft Corporation, Buffalo, New York, has accepted a position as Dynamics Engineer with the Hughes Aircraft Company, Culver City, California.

Miss Joyce B. Friedman, Technical Operations, Inc., Washington, D. C., has been appointed a Member of the Technical Staff with the MITRE Corporation, Bedford, Massachusetts.

Mr. R. A. Gammill, Jr., Philco Corporation, Philadelphia, Pennsylvania, has been appointed Engineer with the Philco Corporation, Willow Grove, Pennsylvania.

Mrs. Carolen M. Jackson, U. S. Air Force Aeronautical Chart and Information Center, St. Louis, Missouri, has accepted a position as Mathematician with the Dugway Proving Ground, Dugway, Utah.

Mr. G. R. Kuhn, Battelle Memorial Institute, has accepted a position as Mathematician with the Bendix Aviation Corporation Research Laboratory, Detroit, Michigan.

Mr. C. P. Lecht, MITRE Corporation, Lexington, Massachusetts, has been appointed Second Lieutenant in the United States Army Ordnance Corps., Aberdeen Proving Ground, Maryland.

Dr. W. W. Leutert, Tidewater Oil Company, Los Angeles, California, has accepted a position as Manager of Advanced Programming and Applied Mathematics with Remington Rand UNIVAC, New York.

Miss Ethelyne L. McBee, Dade County Schools, Miami, Florida, has been appointed Mathematics and Science Teacher at the Tucson Public Schools, Tucson, Arizona.

Mr. William McKay, Franklin Institute, Philadelphia, Pennsylvania, has accepted a position as Research Programmer with the International Business Machines Corporation, White Plains, New York.

Mr. R. H. Miller, University of Minnesota, has been appointed Instructor at Wisconsin State College, Whitewater.

Mr. N. C. Mitrowsis, Fordham University, has accepted a position as Assistant Project Engineer with the Curtiss-Wright Corporation, Wright Aeronautical Division.

Mr. Otto Mond, General Electric Company, Cincinnati, Ohio, has accepted a position as Associate Mathematician with the International Business Machines Corporation, Research Center, Yorktown Heights, New York.

Professor E. P. Northrop will be on leave from the University of Chicago for eighteen months beginning April 1, 1960 to serve as the Ford Foundation's resident representative in Turkey and as Consultant to the Turkish Ministry of Education, with special attention to mathematics and the natural sciences.

Mr. R. R. Parker, HRB—Singer, Inc., State College, Pennsylvania, has accepted a position as Associate Electronic Engineer with Cornell Aeronautical Laboratory Inc., Buffalo, New York.

Mr. Perry Scheinok, Wayne State University, has been promoted to Assistant Professor.

Mr. W. R. Slinkman, Bemidji State College, has been promoted to Assistant Professor.

Mr. C. J. Smith, Washington University, has accepted a position as Methods Technician with Remington Rand UNIVAC, St. Louis, Missouri.

Mr. Irwin Stoner, Service Bureau Corporation, New York, has been promoted to Senior Analyst in the Special Projects Division, Poughkeepsie, New York.

Mr. B. A. Tague, Massachusetts Institute of Technology, has been appointed a Staff Member at Bell Telephone Laboratories, Murray Hill, New Jersey.

Dr. R. S. Varga, Bettis Atomic Power Laboratory, Westinghouse Electric Corporation, Pittsburgh, Pennsylvania, has been appointed Professor at Case Institute of Technology, Cleveland, Ohio.

Miss Janice E. Wilshire, Baylor University, has accepted the position of Mathematician with the Holloman Air Force Base, Alamagordo, New Mexico.

Mr. J. L. Wulff, Sacramento State College, has been promoted to Assistant Professor.

Mr. Rene-Victor Goormaghtigh, La Brugeoise Steelworks, Belgium, died February 10, 1960. He was a member of the Association for fourteen years.

FREE BOOK LIST

The National High School and Junior College Mathematics Club, Mu Alpha Theta, has prepared an annotated bibliography of mathematical books and periodicals suitable for the high school library. The books have been carefully selected to provide a well-rounded library on a limited budget and grouped into six sets of about $30 valuation per

set. The entire library, including all six sets of books and the ten recommended periodicals may be purchased for less than $250.

Any person interested in having a copy may obtain one free by sending a stamped, self-addressed, *number 10 envelope* to: HIGH SCHOOL BOOK LIST, c/o Richard V. Andree, The University of Oklahoma, Norman, Oklahoma.

THE EMPLOYMENT REGISTER

The Mathematical Sciences Employment Register, established by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics, will be maintained at the Summer Meeting in East Lansing, Michigan, on August 29–September 3, 1960. The Register will be conducted from 9:00 A.M. to 5:00 P.M. on August 29–September 2, and from 9:00 A.M. to 12:00 noon on September 3.

The Employment Register Desk will be located in the Snyder Dormitory, the same building in which the Registration Desk will be maintained. Interview tables in communal rooms will be assigned for interviews. There will be no charge for registering to either job applicants or to employers, except when the Late Registration Fee for employers is applicable. Provision will be made for anonymity of applicants upon request.

Job applicants and employers who wish to be listed will please write to the Employment Register, 190 Hope Street, Providence 6, Rhode Island, for application forms and for position description forms, *which must be completed and returned to Providence not later than August 5, 1960, in order to be included free of charge in the listings at the meeting in East Lansing.* Forms which arrive after this closing date, but before August 22, will be included in the listings at the meeting for a Late Registration Fee of $3.00, and will also be included in the sold listings, but not until ten days after the meeting. Printed listings will be available for distribution both during and after the meeting. The prices are as follows: Position descriptions, $2.00; listing of applicants, academic only, $5.00; comprehensive listing of applicants, academic, industrial, and government, $20.00.

It is essential that applicants and employers register at the Employment Register Desk promptly upon arrival at the meeting to facilitate the arrangement of appointments.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor Henry L. Alder, Secretary, announces that the following 267 persons have been elected to membership by the Board of Governors on applications duly certified.

ALFRED T. ANDERSON, M.A. (Michigan) Teacher, Sewanhaka High School, Floral Park, New York

MARC ARONSON, Student, University of Florida

PARVIZ AYUBI, Student, San Jose State College

RICHARD A. ALO, Student, Gannon College

ROBERT H. ANGLIN, Cost Accountant, Dan River Mills, Inc.

DANIEL ASHLER, Ed.M. (Temple) Teacher, School District of Philadelphia

RICHARD H. BALOMENOS, M.A. (N.Y.U.) Teaching Fellow, Graduate School of Education, Harvard

PAUL W. BARCUS, B.S. (M.I.T.) Major, United States Marine Corps

JONATHAN C. BARRON, Student, College of South Jersey, Rutgers, The State University

ROBERT L. BATES, B.A. (Gallaudet Coll.) Senior Mathematician, Dept. of Navy, Washington, D. C.

RICHARD G. BAUER, M.S. (Rutgers) Higgins Fellow, Columbia University

STUART D. BAXTER, Ph.D. (Toronto) Mathematician, National Research Council of Canada

THOMAS L. BEATTY, Student, North Dakota Agricultural College

ALBERT B. BENNETT, JR., M.A. (Maine) Educational Specialist, Dept. of Education, Augusta, Maine

SCOTT A. BENSON, Student, Knox College

JAMES K. BIDWELL, M.S. (Illinois) Teacher, Central High School, Bay City, Michigan

FRANK A. BLOOD, Student, Case Institute of Technology

JAMES R. BOCK, B.A. (San Fernando Valley S.C.) Grad. Student, University of California, Los Angeles

CHARLES H. BOLL, Ph.D. (Stanford) Research Scientist, Lockheed Missile Systems Division

RICHARD G. BRADSHAW, M.A. (Columbia) Asst. Prof., Clarkson College

LINCOLN E. BRAGG, B. S. (Carnegie Inst. Tech.) Grad. Student, Carnegie Institute of Technology

VINCENT BRANT, M.Ed. (Duke) Supervisor, Board of Education of Baltimore County, Maryland

JAMES B. BRENNAN, Student, Fordham University

ROBERT T. BRESNIN, Student, Los Angeles State College

MARJORIE BRICKMAN, B.S. (Creighton) Grad. Asst., University of Nebraska

KENNETH L. BRINKMAN, M.A. (U.C.L.A.) Member, Technical Staff, Hughes Aircraft: Grad. Student, University of California, Los Angeles

MARY S. BROCK, B.S. (Eastern S.C.) Teacher, Harlan High School, Kentucky

BROTHER BENEDICT VIRGIL, B.A. (De LaSalle Normal) Head of Dept., De LaSalle High School, New Orleans, Louisiana

BROTHER L. JOHN, M.A. (DePaul) Head of Chemistry Dept., St. Mel High School, Chicago, Illinois

BROTHER FREDERICK JAMES McGLYNN, B.S. (Notre Dame) Head of Dept., Catholic Central High School, Monroe, Michigan

G. B. BROWN, B.S. (Texas) Instr., Mills College

ANDREW M. BRUCKNER, Ph.D. (U.C.L.A.) Asst. Prof., University of California at Santa Barbara

MARSHALL J. BURGH, Student, Roosevelt University

WILLIAM A. E. BUTLER, Student, Beamsville District High School, Ontario, Canada

F. ANN CALLAHAN, Student, George Pepperdine College

NORMA L. CAMP, Student, Lindenwood College for Women

RICHARD L. CAMPBELL, Student, Case Institute of Technology

JOSEPH F. CARLS, Student, St. Bonaventure University

J. W. CARMICHAEL, Jr., Student, Western New Mexico University

HENRY E. CASTRO, B.S.E.E. (Texas) Research Scientist, Applied Physics Laboratory, University of Texas

WILLIAM L. CAUDLE, Student, New Mexico State University

ANSON E. CHAPMAN, Student, Roosevelt University

HARRY S. P. CHIN, M.S. (Rensselaer Polytech. Inst.) Senior Mathematician, Bendix Pacific

YIU HUNG CHAN, Student, Queen's University

VICTOR CH'IU, Student, Kent State University

CHANG CHOI, Student, The University of the South

FRANCIS J. COLLINS, Ed.M. (Boston S.T.C.) Head of Dept., Natick High School, Massachusetts

WILLIAM C. CONNETT V, Student, Georgetown University

F. JOE CROSSWHITE, M.Ed. (Missouri) Instr., Keokuk Community College

MRS. ANNE W. CUNNINGHAM, A.M. (Michigan) Teacher, Ann Arbor Senior High School, Michigan

DOYLE O. CUTLER, B.A. (Texas Christian) Grad. Student, Texas Christian University

ROBERT F. DAIGNEAULT, Student, Lebanon Valley College

RICHARD H. DALE, B. S. (Maryland) Mathematician, White Sands Missile Range

JAMES W. DEEGAN, Student, St. Bonaventure University

RUSSELL E. DENMAN, B.S.E.E. (Purdue) Engr., Lockheed Missile Systems Division

CALVIN DESPORTES, B.S.E.E. (Georgia Inst. Tech.) Retired, Apalachicola, Florida

HERBERT C. DIMLICH, B.S. (Boston) Head of Dept., West High School, Pawtucket, Rhode Island

RAYMOND J. DISTLER, M.S. (Kentucky) Instr., College of Engineering, University of Kentucky

JAMES S. DOMBEK, B.S.E. (Arkansas S.T.C.) Asst. Instr., University of Kansas

JOHN P. DOWNES, M.A. (New York S.T.C., Albany) Asst. Prof., State University of New York, College of Education at Oneonta

KLAUS E. ELDRIDGE, Student, Hardin-Simmons University

WILLIAM R. EMERSON, Student, California Institute of Technology

ROBERT M. EWART, Jr., Student, Case Institute of Technology

MRS. BESSIE V. FAIRLEY, M.A. (Southwestern Louisiana Inst.) Head of Dept., Opelousas High School, Louisiana

LEONARD M. FALTZ, Student, College of the City of New York

RICHARD A. FELLOWS, B.S. (San Jose S.C.) Statistician, Fairchild Semiconductor Company

ARTHUR C. FLECK, B.S. (Western Michigan) Grad. Asst., Michigan State University

CHUNG FONG, B.A. (California) Grad. Student, Sacramento State College

RICHARD D. FREEMAN, JR., Student, Michigan State University

FRANCIS R. FROLA, B.A. (San Jose S.C.) Grad. Asst., Washington University

GREGORIO FUENTES, Licenciado (Madrid) Instr., Gonzaga University

CHARLES D. GALLANT, Student, St. Francis Xavier University

PAUL J. GAMBAIANA, B.A. (Iowa S.T.C.) Chairman of Dept., West Sioux High School, Hawarden, Iowa

JOHN B. GARNETT, Student, University of Notre Dame

N. HUGH GEARY, Student, Purdue University

ABOLGHASSEM GHAFFARI, D.Sc. (Sorbonne), Ph.D. (London) Aeronautical Research Scientist, National Bureau of Standards, Washington, D. C.

DENNIS M. GIRARD, Student, University of Detroit

EUGENE C. GLUESING, M.A. (Minnesota) Advanced Programmer, Remington Rand UNIVAC

JEROME X. GOLDSCHMIDT, B.S. (Dayton) NDA Fellow, Michigan State University

MARGARET J. GORDON, Student, Cornell University

JOAN A. GOTTLIEB, Student, Barnard College, Columbia University

BARRY GRANOFF, Student, Fairleigh Dickinson University

GEORGE L. GROSS, Ph.D. (Iowa S.U.) Lecturer, Adelphi College: Staff Consultant, Grumman Aircraft Engineering Corporation

FRANK A. GUGINO, M.E. (Rensselaer Polytech. Inst.) Consulting Engr., Tonawanda, New York

PHIL W. HAGEN, Student, St. John's University

JAMES E. HALL, A.M. (Harvard) Instr., Northern Illinois University

JOHN H. HANCOCK, M.A. (Southern California) Senior Mathematician, Marquardt Corporation

LAWRENCE E. HANSON, M.S. (California, Davis) Instr., Chico State College

WESLEY E. HARTMAN, Student, University of California, Los Angeles

PAUL M. HASSE, B.S. (S.U. South Dakota) Junior Statistician, North American Aviation

WILLIAM S. HATCHER, M.A. (Vanderbilt) Teaching Asst., Vanderbilt University

REV. ROBERT A. HAUS, M.S. (Notre Dame) Instr., Canisius College

JOHN C. HAWKINS Jr., M.B.A. (Harvard) Teacher, Hastings High School, Hastings-on-Hudson, New York

CHARLES F. HICKS, M.A. (Polytech. Inst. Brooklyn) Senior Structural Engr., Republic Aviation Corporation

ROGER L. HIGDEM, M.S. (North Dakota) Asst. Prof., College of Idaho

LEONA L. HIRZEL, M.A. (Columbia) Asso. Prof., State University of New York, College of Education at Oneonta

DENNIS L. HOBDE, Student, Sacramento State College

MRS. BELVA H. HOPKINS, B.A. (Western Maryland Coll.) Teacher, High Point Senior High School, Beltsville, Maryland

WILBUR F. HOPPE, M.S. (Wisconsin) Instr., Western Illinois University
ROBERT G. HOSEA, M.A. (Stanford) Teacher, Rosemead High School, California
WILLIAM H. HOUPT, B.A. (Dickinson Coll.) Engr., Western Electric Company
JOSEPH T. HOWSON, JR., B.S. (Rensselaer Polytech. Inst.) Grad. Asst., Rensselaer Polytechnic Institute
FRANCIS M. HUDSON, M.A. (Texas) Instr., Western State College
BRUCE K. HURLBERT, M.A. (Long Beach S.C.) Instr., Mt. San Antonio Junior College
CLINT L. HYATT, M.A. (Louisiana S.U.) Auditor, Ponca City, Oklahoma
LARON J. JACOBSEN, Student, University of Utah
ROBERT C. JAMES, JR., Student, Texas Christian University
THOMAS M. JENKINS, A.B. (Kenyon Coll.) Asst. in Instruction & Grad. Student, Yale University
MRS. DORRIS H. JOHNSON, M.A. (Florida) Chairman of Dept., Clearwater High School, Florida
PAUL H. JOHNSON, Aircraft worker, Convair
ROBERT A. JOHNSON, B.S. (Iowa S.C.) Engr., Lockheed Aircraft Corporation
ROBERT W. JOHNSON, B.A. (Amherst Coll.) Grad. Student, Massachusetts Institute of Technology
WARD D. JONES, M.B.A. (Harvard) Grad. Student, American University
JAMES P. KALLENBORN, Student, Kent State University
DWIGHT E. KECK, B.S.Ed. (Ball S.T.C.) Grad. Asst., Michigan State University
JOHN F. KEPLER, M.A. (Vanderbilt) Grad. Student, Vanderbilt University
IRVING J. KESSLER, Student, Brooklyn College
GENE A. KEMPER, M.S. (North Dakota) Instr., University of North Dakota
LAEL F. KINCH, Student, University of Kentucky
JOHN R. KING, M.A. (Boston Coll.) Asst. Prof., Providence College
DAVID E. KOLTENBAH, Student, Kent State University
LAWRENCE J. KRATZ, Student, Xavier University
HUDSON VANE. KRONK, B.S. (Rensselaer Polytech. Inst.) Grad. Asst., Michigan State University
ROBERT L. KRUSE, Student, Pomona College
JOHN P. LAMB, Student, University of Notre Dame
CAPT. WILLIAM R. LAMBERT, M.S. (Colorado) Instr., United States Air Force Academy
RICHARD B. LANE, Student, Carleton College
JOHN D. LAWRENCE, Student, Washington State University
JOSEPH R. LEROY, M.S. (Alabama) Senior Technologist, Shell Chemical Company
RAYMOND P. LEROY SR., B.S. (Troy S.C.) Head Coach & Instr., Hayneville High School, Alabama
MARTIN LEVY, A.B. (Sacramento S.C.) Lecturer, Sacramento State College
GENE L. LEWIS, Student, Harvard University
KARL G. LIEBLONG, JR., B.S.Ed. (Westchester S.T.C.) Head of Dept., Collingdale High School, Pennsylvania
GEORGE E. LINDAMOOD, Student, Wittenberg University
EDWARD B. LIPTON, M.A. (Columbia) Mathematical Analyst, Lockheed Aircraft Corporation
BETTY LIDE, M.A. (North Carolina) Instr., Temple University
JAMES B. LISK, A.B. (Northwest Nazarene) Instr., Columbia Basin College
GEORGE W. LOFQUIST, M.Ed. (North Carolina) Instr., Louisiana State University in New Orleans
WILLIAM A. LOPEZ, Student, LeMoyne College
MARY LOU LYONS, Student, University of Kentucky
CHARLES R. MACCLUER, Student, Ohio State University
NEIL A. MACKIE, Student, St. Joseph's College
ROBERT A. MACLEOD, Student, University of Alberta
JAMES T. MADEJ, Student, KNOX College
JOHN MAHONEY, M.A., (Ohio S.U.), Instr., Yuba College
FREDERICK R. MAIOCCO, Student, West Virginia University
FRED MARER, M.A. (Southern California) Chairman of Dept., Los Angeles City College
DUDLEY R. MARSH, B.S. (Massachusetts) Instr., Worcester Junior College
MORRIS L. MARX, B.S. (Tulane) Grad. Student, Tulane University
CHARLES E. MAYO, Student, Croft High School, Waterbury, Connecticut
DAVID A. MCBLAIN, Student, Grinnell College
LOUIS R. MCCREERY, B.A. (U.C.L.A.) Head of Dept., San Marino High School, California
ELLIS R. MCDANIELS, Student, University of Wichita
BOBBY K. MCGUIRE, Student, North Central College
ERNEST E. MCINTOSH, JR., Student, San Francisco State College
ANTHONY J. MCKEE, Student, St. Bonaventure University
ROBERT W. MEANS, Student, University of Santa Clara
ROBERT K. MEANY, Ph.D. (Wisconsin) Asst. Prof., Texas Christian University
JOSEPH P. MEISSNER, Student, Xavier University
RUTH C. MERKLE, M.A. (Western Reserve) Chairman of Dept., Avon Lake High School, Ohio
JOHN R. METZNER, JR., B.S. (Bethany Coll.) Grad. Asst., Pennsylvania State University
JAMES E. MILLER, A.B. (Western Kentucky S.C.) Teaching Fellow, University of Kentucky
JERALD E. MILLER, B.S. (Ball S.T.C.) Teacher, North Side High School, Fort Wayne, Indiana
RAY MINES III, Student, University of Washington
CHARLES O. MINNICH, M.A. (Claremont Grad. School) Instr., Mt. San Antonio College
JOHN W. MOON, B.A. (Bethany Nazarene Coll.) Grad. Asst., Michigan State University
MELLICENT S. MOORHEAD, Ph.D. (Pittsburgh) Asso. Prof., Gettysburg College
ALLEN L. MORTON, JR., M.A. (Yale) Mathematician, I.B.M., New York City
JAMES B. MUNZ, M.A. (Syracuse) Asso. Prof., LeMoyne College
MAURICE R. NELSON, Student, San Jose State College
PAUL A. NICKEL, Ph.D. (U.C.L.A.) Asst. Prof., Montana State College
WILLIAM R. NIPPER, B.A. (San Jose S.C.) Teacher, Mariposa County High School, California
REV. TIMOTHY NORMAN, B.A. (St. Procopius Coll.) Asst. Head of Dept., Marmion Military Academy, Aurora, Illinois
THOMAS V. O'BRIEN, B.S. (Xavier) Grad. Student, Xavier University
REV. THOMAS OSTDICK, PH.D. (Notre Dame) Instr., St. Meinrad College
JAMES PACHARES, Ph.D. (North Carolina) Asst. Section Head, Hughes Aircraft Company
ALAN H. PAINE, M.A. (Columbia) Asso. Prof., Potomac State College
BERNARD L. PARDEE, M.A. (Drake) Chairman of Dept., El Camino High School, Sacramento, California
NICK PASSELL, Student, Swarthmore College
MAGDALENE R. PATTERSON, B.S. (Bradley) Teacher, Charlestown High School, Indiana
JAMES T. PEPE, Student, Xaverian High School, Brooklyn, New York
ALBERT D. POLIMENI, Student, University of Buffalo
MRS. MARGARET M. POOL, M.S. (Louisiana S.U.) Instr., Louisiana State University at New Orleans
ANDRÉ G. POUX, Student, University of Detroit
THOMAS G. PRICE, Student, West Virginia University
MRS. SHIRLEY S. REBA, M.S. (Arizona) Teacher, Glenbrook High School, Northbrook, Illinois
HAZEL JO REED, Student, Reed College
ALLAN S. REHM, M.S. (Illinois) Asst., University of Illinois
MRS. MARIE L. RENAUD, M.A. (Tulane) Teacher, Fortier Senior High School, New Orleans, Louisiana
DANIEL L. RIBAR, Student, Xavier University
NORMAN M. RICE, Student, Queen's University
MRS. THELMA A. RICE, M.S. (Alabama S.C.) Asso. Prof., Alabama State College
MAY E. RICHARDS, M.A. (Illinois) Chairman of Dept., Glenbrook High School, Northbrook, Illinois
EARLE W. RICHARDSON, JR., Student, American University
JOHN W. ROSBRUGH, Student, Southwest Missouri State College
HENRY ROSENBERG, Student, Brooklyn College
HERBERT RUDERFER, M.A. (Wisconsin) Mathematician, I.B.M., New York City

JOHN T. RUPLEY, Student, Xavier University
HOWARD C. SAAR, B.S. (Illinois State Normal) Teacher, Normal Community High School, Illinois
EDWIN R. SAGE II, B.A. (Williams Coll.) Teacher, Middlesex School, Concord, Massachusetts
MRS. SHU-SHEN SAH, M.S. (Illinois) Research Asst., University of Illinois
MARGARET E. SALMON, Student, Pomona College
GERALD M. SAMSON, A.M. (Michigan) Instr., Flint Junior College
NILDA G. SANTIAGO, B.S. (Puerto Rico) Instr., University High School, Rio Piedras, Puerto Rico
BRUCE D. SCHAEFFER, Student, Albright College
JOSEPH T. SCHARTMAN, B.S. (Fenn Coll.) Engineering Analyst, Thompson, Ramo-Wooldridge
JOSEPH F. SCHINDLER, M.S. (Oklahoma S.U.) Asst. Prof., Clemson College
ROBERT J. SCHWABAUER, M.S. (Nebraska) Asst., University of Nebraska
BRITT J. SCHWEITZER, M.S. (Northwestern) Asst. Prof., San Jose State College
JAMES L. SHAWN, JR., M.S. (North Texas S.C.) Acting Head of Dept., Arlington State College
HAZEL L. SHELTON, M.A. (Missouri) Teacher, Carrollton High School, Missouri
RICHARD E. SHEPARDSON, Student, LeMoyne College
DAVID G. SHIELDS, Student, Fenn College
WILLIAM H. SHY, M.A. (Georgia) Research Analyst, Kimberly-Clark Corporation
CARL D. SIKKEMA, B.A. (Calvin Coll.) Grad. Asst., University of Miami
CARL D. SIKKEMA, B.A. (Calvin Coll.) Grad. Asst., University of Miami
JOHN J. SILVERSON, B.S. (Indiana Tech. Coll.) Programmer/Operator, RCA Service Company
SISTER BERNARDINE MARIE, M.A. (DePaul) Teacher, Aquinas Dominican High School, Chicago, Illinois
SISTER MARIAN JOAN MCKENNA, M.A. (Detroit) Chairman of Dept., Holy Names College
SISTER MARIE CHRISTINE, B.S. Teacher, Our Lady of the Rosary, Philadelphia, Pennsylvania
SISTER MARIS STELLA SCHROT, B.A. (Immaculate Heart Coll.) Grad. Student, University of Notre Dame
SISTER MARY ALPHONSUS, A.B. (Aquinas Coll.) Grad. Student, University of Notre Dame
SISTER MARY MECHTILDE HILL, Ph.D. (St. Louis) Chairman of Dept., College of St. Mary
SISTER MARY MYLES, B.A. (Mt. St. Mary Coll.) Teacher, St. Joseph High School, Manchester, New Hampshire
SISTER MARY PATRICK COLLINS, B.A. (Coll. St. Rose) Teacher, Keveny Academy, Cohoes, New York
SISTER PAUL JAMES, Ph.D. (Notre Dame) Instr., Barry College
DENIS K. SJERVE, Student, University of British Columbia
CHARLES E. SKINNER, JR., B.S. (Hope Coll.) Grad. Asst., Michigan State University
CECIL B. SMITH, Student, Mississippi Southern College
WILLIAM B. SMITH, B.S. (Lamar S.C., Tech.) NDA Fellow, Agricultural and Mechanical College of Texas
EDWARD L. SPITZNAGEL, JR., Student, Xavier University
SELIG STARR, M.A. (George Washington) Mathematician, Dept. of the Army, Washington, D.C.
W. ASHMORE STEED, Student, Georgia Institute of Technology
RONALD S. STEEVES, Student, Trinity College
COLEMAN L. STERLING, M.Ed. (Texas Southern) Asst. Prof., Grambling College
DALE R. STEVENS, M.A. (Michigan S.U.) Teacher, Chairman of Dept., Upland High School, California
AARON S. STRAUSS, Student, Case Institute of Technology
RICHARD E. STRENK, Student, Xavier University
JOSEPH E. SWEENEY, Student, St. Francis Xavier University
WILLIAM J. SWEENEY, JR., Student, University of Notre Dame
WILLIAM SWYTER, M.A. (Bob Jones) Instr., Montgomery Junior College
ROBERT TAYLOR, Student, Brooklyn College
H. LAVERNE THOMAS, M.A. (Syracuse) Asso. Prof., State University of New York, College of Education at Oneonta
RUSSELL E. THOMPSON, M.A. (South Carolina) Asst. Prof., The Citadel
DONALD W. TRASHER, B.A. (Houghton Coll.) Teaching Fellow, University of Buffalo
STEPHEN V. ULLOM, Student, The American University
ALBERT R. VANCLEAVE, JR., B.S. (Alabama Coll.) Teacher, Homewood Junior High Alabama
THOMAS C. VANFLANDERN, Student, Xavier University
FLOYD R. VEST, M.A. (Oklahoma) Instr., East Texas State College
CARL L. WAAG, Asst. Auditor, Hotel Taft, New York City
JOHN H. WADDELL III, Ph.D. (Michigan) Astronomer, Sacramento Peak Observatory
EDWARD T. WALKER, M.A. (Indiana) Instr., Muskingum College
MARGARET E. WALTERS, Student, University of Kentucky
JACK K. WARD, M.A. (San Jose S.C.) Teacher, Fairfield Joint Union High School, California
LEON J. WEILL, B.A. (Mississippi Southern Coll.) Grad. Fellow, Mississippi Southern College
ROBERT J. WEISS, B.A. (LaVerne Coll.) Grad. Student, University of California, Los Angeles
CHARLES W. WELSH, Student, St. Bonaventure University: Manager, Empire Gas Filling Stations
BURDETTE C. WHEATON, B.S. (Mankato) Grad. Asst., State University of Iowa
SHIRLEY M. WIEMERS, B.S. (St. Mary's Dominican Coll.) Teacher, St. Mary's Dominican High School, New Orleans, Louisiana
NORMAN F. WILLIAMSON, JR., M.S. (Tulane) Instr., Louisiana State University at New Orleans
RALPH A. WILTSE, Student, Central Michigan University
THOMAS W. WINCHESTER, Student, Ursinus College
CHARLES A. WINFIEL, Student, Southwest Missouri State College
ANTHONY J. WINKLER, Student, Agricultural and Mechanical College of Texas
SAMUEL WISHLOFF, B.Ed. (Alberta) Teacher, Westlock High School, Alberta, Canada
HUNG-HSI WU, Student, Columbia University
KENNETH P. YANOSKO, Student, Xavier University
MRS. MARY H. ZINK, M.S. (S.U. Iowa) Chairman of Dept., Elmhurst College

THE FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The thirty-seventh annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at the Buena Vista Hotel, Biloxi, Mississippi on February 19–20, 1960 with Centenary College acting as host institution. The Friday afternoon session was held in two concurrent sessions. Professor W. M. Sanders, Mississippi Vice-Chairman and Professor T. K. Maddox, Louisiana Vice-Chairman, presided. Professor S. B. Murray of Mississippi State University, Chairman of the Section, presided at the Friday evening and Saturday morning sessions. There were 162 persons registered, including 75 members of the Association.

The following officers were elected for the coming year: Chairman, Professor Z. L. Loflin, Southwestern Louisiana Institute; Vice-Chairman for Louisiana, Professor J. H. Wahab, Louisiana State University at New Orleans; Mississippi Vice-Chairman, Pro-

fessor Eleanor Walters, Delta State College; Secretary-Treasurer, Professor T. L. Reynolds, Millsaps College.

Reports on the High School Mathematics Contest were given by Professor H. T. Karnes of Louisiana State University and Professor N. A. Childress of the University of Mississippi showing increased participation by the high schools. The section instructed these chairmen to extend invitations to all schools in both states next year.

The invited speaker for the meeting was Professor F. E. Ulrich of The Rice Institute. He presented the two following addresses:

1. *Non-Euclidean metrics from the standpoint of Cayley-Klein-Poincaré and their use in studying properties of analytic functions.*

After general remarks concerning geometries defined by subgroups of the projective group, the Cayley-Klein definition of distance and angle was given and the three classical geometries defined. The Klein model for the hyperbolic case was discussed and from it the Poincaré model obtained. The rigid motions were then expressed as nonsingular linear transformations of a complex variable. The interpretation of Schwarz's lemma in this metric was given and Pick's formulation was discussed. It was shown that this metric may also be defined by means of the differential form $|dz|/(1-|z|^2)$. More general metrics were considered and their role in the Riemann surface type problem indicated.

2. *The behavior of functions harmonic and positive in an angular domain.*

Functions $u(x, y)$ were considered which are harmonic and positive in a simply-connected domain D and zero on the boundary B of D except at one point O. The function must become infinite as $(x, y)\to O$ if $u(x, y)\not\equiv 0$. Theorems were considered which relate the growth of $u(x, y)$ as $(x, y)\to O$ in D to the character of the boundary B in the neighborhood of O. The general situation is: The more pointed D is in the neighborhood of O, the more rapidly $u(x, y)$ becomes infinite. If the boundary has a cusp at O, $u(x, y)$ becomes infinite more rapidly than any power of $1/r$ where r is the distance from (x, y) to O. The precision which may be obtained depends on the regularity of B in the neighborhood of O.

Other papers presented are as follows:

3. *The Undergraduate Honors Program at Louisiana State University*, by Professor R. D. Anderson, Louisiana State University.

The methods of choosing the superior student in mathematics and the courses taught for these students were discussed, together with the impact of the undergraduate honors program on the graduate program.

4. *The National Science Foundation summer program for high school juniors*, by Professor H. S. Butts, Louisiana State University, introduced by the Secretary.

The type of student that took part in the National Science Foundation summer program for high school juniors at Louisiana State University in 1959 and the type of mathematics courses designed for these students were discussed, together with the objectives of such a course.

5. *Two geometrical interpretations of the scalar product of two vectors*, by Professor B. E. Mitchell, Belhaven College.

Let $\mathbf{a}$ and $\mathbf{b}$ be two positional vectors, common origin O, termini A and B respectively. *Theorem* I: The scalar product $\mathbf{a}\cdot\mathbf{b}$ is the power of O with respect to the circle constructed on AB as a diameter. *Theorem* II: The product $\mathbf{a}\cdot\mathbf{b}$ is the radius of a circle, center O, with respect to which A or B and the projection of B on OA, or the projection of A on OB are inverse points.

6. *Semigroups in introductory modern algebra*, by Professor T. F. Mulcrone, S.J., Loyola University.

In view of the growing prominence of the theory of semigroups and the more frequent ref-

erences to them in group, ring and field theory, the author suggests that the treatment of groups in introductory modern algebra courses might profitably be preceded by the definition of the semigroup and the use of examples of semigroups to illustrate the closure, associativity, identity and inverse laws. Several elementary semigroups were given to illustrate these laws and to provide examples of such elements as zeros, idempotents, nilpotents and ideals. The importance of the semigroup of the totality of transformations (mappings) of a set into itself was pointed out.

7. *Non-Euclidean numbers*, by Professor E. B. Moyers, University of Mississippi, introduced by the Secretary.

That the postulates of a non-Euclidean geometry form the basis of a consistent logical system is shown by comparison with the set of real numbers that have the property of being less in absolute value than a fixed positive number. To facilitate the discussion these numbers are shown to form a field under operations defined in terms of elementary functions.

8. *On properties weaker than countable compactness*, by Mr. Leon Weill, Mississippi Southern College, introduced by the Secretary.

In a completely regular space X, several statements were shown to be equivalent.

9. *On convergence properties in $C(X)$ equivalent to $C(X)=C^*(X)$*, by Mr.C. T. Hathorn, Mississippi Southern College, introduced by the Secretary.

The equivalence of several statements in any topological space and a completely regular space were considered.

10. *Pseudo-compact spaces*, by Professor R. W. Bagley, Mississippi Southern College.

Some recent results concerning pseudo-compact spaces are discussed. In particular results concerning product spaces and those concerning convergences in $C(X)$ are considered.

11. *Quantum statistics*, by Professor Margaret M. LaSalle, Southwestern Louisiana Institute.

This paper presents the fundamental reason for quantum statistical theory. It is shown by rules for independent probabilities, linear combinations of the wave functions $\sum C_i\psi_i$ and by the probability of the set of eigenvalues $|C_i|^2$, that in general two or more wave functions are not equivalent to a single wave function. Since quantum statistical mechanics has to do with many quantum mechanical states, measures of previous states and expected future states are essential.

12. *Continuous solutions of a functional equation*, by Professor R. D. Boswell, Jr., Mississippi State University.

The author discusses the continuous real-valued functions which satisfy the equation $f(x+y)=f(x)[1+af^2(y)]^{1/2}+f(y)[1+af^2(x)]^{1/2}$, where a is a real number.

13. *On solving systems of linear equations*, by Professor B. E. Mitchell, Louisiana State University.

A method similar to Chio's pivotal condensations method for evaluating determinants for matrices yields a method for solving systems of linear equations similar to Crout's method.

T. L. REYNOLDS, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The fortieth regular meeting of the Southern California Section of the Mathematical Association of America was held at Los Angeles State College, Los Angeles, California, on March 12, 1960. Professor D. H. Hyers, Chairman of the Section, presided. The registered attendance was 150, including 118 members of the Association.

At the business meeting Professor J. D. Swift, Chairman of the Nominating Committee, reported that the following officers were elected by mail ballot for the next academic

year: Chairman, Professor R. C. James, Harvey Mudd College; Vice-Chairman, Professor Clifford Bell, University of California, Los Angeles; Secretary-Treasurer, Mr. R. B. Herrera, Los Angeles City College. Professor Swift reported also that the members of the Program Committee for the coming year were: Professors P. A. White, University of Southern California, chairman; C. A. Seekins, Occidental College; R. M. Redheffer, University of California, Los Angeles; P. J. Kelly, University of California, Santa Barbara; L. T. Black, Long Beach City College.

The following program was presented:

1. *Devil's curve and witch of Agnesi*, by Mr. T. E. Sydnor, Pasadena City College.

Research indicates that the Devil's curve got its name from a resemblance to a spindle shaped top called a diabolo, popular as a toy in the early twentieth century. The speaker exhibited a diabolo and indicated how the graph of the Devil's curve may easily be sketched. To honor the scholarly humanitarian Agnesi, the speaker suggested that the so called witch be known henceforth as the curve of Agnesi.

2. *A property of tetrahedrons*, by Professor Morgan Ward, California Institute of Technology.

The speaker showed that the triangle inequality for the chordal metric in the complex plane used by Carathéodory in his *Theory of Functions* (New York, 1954) is implied by a theorem of plane geometry stating that the product of the diagonals of a plane quadrilateral is less than or equal to the sum of the products of the pairs of opposite sides. A simple proof was given of the latter theorem; the case of equality is Ptolemy's theorem on quadrilaterals inscribed in a circle.

3. *A problem in communication theory*, by Dr. A. V. Balakrishnan, Space Technology Laboratories, introduced by the Secretary.

The following problem was discussed: Given N points on the unit sphere in Euclidean space, dimension D, find the distribution of points which maximizes the minimum distance. For $D=2$, one divides the circumference into N equal parts. For $D=3$, solutions have been given for $N<9$ and $N=12$, by Schutte and Van Der Waerden (*Math. Ann.*, 1951), most of the solutions being semiregular polyhedra. For general N and D, one has the theorem: *If* $D \geqq N-1$, *then the optimal polyhedron is the* $N-1$ *dimensional regular simplex with* N *vertices, and* $d_{\min}=\sqrt{(2N/(N-1))}$. In Hilbert space ($D$ and N both $+\infty$) the solution is any set of orthonormal unit vectors, and $d_{\min}=\sqrt{2}$.

4. *Finite projective planes*, by Professor Marshall Hall, Jr., California Institute of Technology.

A finite projective plane is said to be of order n if a line contains exactly $n+1$ points. Every known finite plane is of prime power order. The speaker discussed the known types of finite planes. The fact that the conjectures of Euler and Mann on orthogonal squares are now known to be false might mean that the Bruck-Ryser necessary conditions for the existence of a finite plane are also sufficient.

5. *Regular graphs*, by Professor D. M. Merriell, University of California, Santa Barbara.

The graphs considered were nonoriented finite graphs, with each two vertices joined by at most one arc, and no vertex joining itself. A graph is regular if every vertex has the same degree, the degree being the number of arcs which meet at the vertex. Properties of regular graphs were discussed, including possible degrees for a given order and a criterion for isomorphism. Some unsolved problems and conjectures were stated.

6. *Evaluation of double integrals*, by Professor G. B. Price, California Institute of Technology and the University of Kansas.

The Fundamental Theorem of the Integral Calculus states that $\int_a^b f'(x)dx=f(b)-f(a)$. The usual corresponding theorem in two dimensions states that a double integral can be evaluated by

iterated integrals, but the results in the two cases do not appear to be analogous. This paper contains a treatment of the evaluation of double integrals by methods which are entirely similar to those used in the one dimensional case. Several standard formulas appear as special cases of a general theory.

7. *The Pasadena experiment with School Mathematics Study Group materials*, by Mr. W. H. Glenn, Jr., Mathematics Coordinator, Pasadena City Schools.

Pasadena teachers have been experimenting for two years with School Mathematics Study Group materials for seventh- and eighth-grade mathematics, as one of several groups financed by the National Science Foundation. The general philosophy has been "to think of grades 7 and 8 not as the end of elementary school mathematics, but rather as a foundation for the work of the senior high school." Professor Richard Dean, California Institute of Technology, has served as the local consultant. The test centers have been evaluating the materials while teaching them in the classroom, with an ultimate goal of developing improved textbook material.

R. B. HERRERA, *Secretary*

OFFICERS AND COMMITTEES AS OF MAY 1, 1960

OFFICERS

President, C. B. ALLENDOERFER, University of Washington (1959–1960)
First Vice-President, A. S. HOUSEHOLDER, Oak Ridge National Laboratory (1960–1961)
Second Vice-President, HARLEY FLANDERS, University of California, Berkeley (1959–1960)
Editor, R. D. JAMES, University of British Columbia (1957–1961)
Secretary, H. L. ALDER, University of California, Davis (1960–1964)
Treasurer, H. M. GEHMAN, University of Buffalo (1958–1962)
Associate Secretary, L. J. MONTZINGO, JR., University of Buffalo (1958–1962)

ADDITIONAL MEMBERS OF THE BOARD OF GOVERNORS

Ex-Presidents

E. J. MCSHANE, University of Virginia (1955–1960)
W. L. DUREN, Jr., University of Virginia (1957–1962)
G. B. PRICE, University of Kansas (1959–1964)

Governors at Large

HOWARD EVES, University of Maine (1958–1960)
J. S. FRAME, Michigan State University (1958–1960)
R. V. CHURCHILL, University of Michigan (1959–1961)
MORRIS KLINE, New York University (1959–1961)
R. C. BUCK, University of Wisconsin (1960–1962)
J. G. KEMENY, Dartmouth College (1960–1962)

Sectional Governors (July 1, 1957–June 30, 1960)

Allegheny Mountain, J. C. KNIPP, University of Pittsburgh
Indiana, LAMBERTO CESARI, Purdue University
Kentucky, R. S. PARK, Eastern Kentucky State College
Metropolitan New York, JEWELL H. BUSHEY, Hunter College
Nebraska, W. G. LEAVITT, University of Nebraska
Northern California, GEORGE PÓLYA, Stanford University
Oklahoma, W. N. HUFF, University of Oklahoma
Rocky Mountain, C. R. WYLIE, JR., University of Utah
Wisconsin, R. D. WAGNER, University of Wisconsin

Sectional Governors (July 1, 1958–June 30, 1961)

Kansas, R. G. SMITH, Kansas State College, Pittsburg
Missouri, W. R. UTZ, JR., University of Missouri
New Jersey, WILLIAM FELLER, Princeton University
Northeastern, F. M. STEWART, Brown University
Ohio, G. M. MERRIMAN, University of Cincinnati
Pacific Northwest, A. T. LONSETH, Oregon State College
Southeastern, G. B. HUFF, University of Georgia
Southwestern, CHARLES WEXLER, Arizona State University
Upper New York State, H. S. M. COXETER, University of Toronto

Sectional Governors (July 1, 1959–June 30, 1962)

Illinois, ROTHWELL STEPHENS, Knox College
Iowa, H. T. MUHLY, State University of Iowa
Louisiana-Mississippi, ARTHUR OLLIVIER, Mississippi State College
Maryland-D.C.-Virginia, R. C. YATES, College of William and Mary
Michigan, R. M. THRALL, University of Michigan
Minnesota, J. M. H. OLMSTED, University of Minnesota
Philadelphia, ALBERT WILANSKY, Lehigh University
Southern California, P. B. JOHNSON, University of California, Los Angeles
Texas, W. T. GUY, JR., University of Texas

COMMITTEES OF THE ASSOCIATION*

ADVISORY COMMITTEE FOR A SURVEY OF NON-TEACHING MATHEMATICAL EMPLOYMENT

MORRIS OSTROFSKY, *Chairman;* PAUL ARMER, T. E. CAYWOOD, CHURCHILL EISENHART, WALLACE GIVENS, Z. I. MOSESSON, G. B. THOMAS.

COMMITTEE ON EARLE RAYMOND HEDRICK LECTURES

L. H. LOOMIS, *Chairman* (1958–1960); WILLIAM FELLER (1960–1962), A. S. HOUSEHOLDER (1959–1961).

COMMITTEE ON HIGH SCHOOL CONTESTS

D. B. LLOYD, *Chairman* (1959–1961); H. M. BACON (1959–1961), R. C. BUCK (1959–1961), A. J. COLEMAN (1960–1962), W. H. FAGERSTROM (1959–1961), E. D. NICHOLS (1959–1960), C. T. SALKIND (1959–1960), L. F. SCHOLL (1957–1960), ARNOLD WENDT (1957–1960).

COMMITTEE ON HIGH SCHOOL TEACHERS' CONTEST

HARLEY FLANDERS, *Chairman;* P. S. JONES, E. E. MOISE, R. E. K. ROURKE, H. W. SYER.

COMMITTEE ON INSTITUTES

E. A. CAMERON, *Chairman;* E. G. BEGLE, W. T. GUY, JR., K. O. MAY, W. H. L. MEYER.

COMMITTEE ON PRODUCTION OF FILMS

L. W. COHEN, *Chairman;* C. B. ALLENDOERFER, E. G. BEGLE, GEORGE SPRINGER, R. L. WILDER.

COMMITTEE ON PUBLICATIONS

ROY DUBISCH, *Chairman;* H. M. GEHMAN, D. H. HYERS, RALPH D. JAMES, ROBERT C. JAMES, J. R. MAYOR.

---

* Terms of office of members expire, except where otherwise noted, at the annual meeting in January following the last year of service listed below. For temporary committees no terms of office are listed, since they are automatically discharged at the expiration of the President's term of office which is at the annual meeting in January 1961.

COMMITTEE ON SECONDARY SCHOOL LECTURERS

J. R. MAYOR, *Chairman* (1958–1960); ROY DUBISCH (1958–1960), W. E. FERGUSON (1958–1961), F. A. FICKEN (1958–1961), H. T. KARNES (1958–1960), C. O. OAKLEY (1960–1962), MRS. MARIE S. WILCOX (1958–1961).

COMMITTEE ON SECTIONS

L. J. MONTZINGO, JR. (*ex officio*), *Chairman;* I. L. BATTIN (1959–1962), ROY DUBISCH (1957–1960), A. W. MCGAUGHEY (1959–1963), H. A. ROBINSON (1959–1961).

COMMITTEE ON SLAUGHT MEMORIAL PAPERS

R. C. BUCK, *Chairman* (1958–1960); F. A. FICKEN (1959–1961), W. T. GUY, JR., (1960–1962), R. D. JAMES, *ex officio.*

COMMITTEE ON THE AWARD FOR DISTINGUISHED SERVICE TO MATHEMATICS

WALLACE GIVENS, *Chairman* (1960–1961); J. W. GREEN (1960), R. J. WALKER (1960–1962).

COMMITTEE ON THE EMPLOYMENT REGISTER

A. E. TAYLOR, *Chairman* (1958–1960, MAA); R. M. THRALL (1959–1961, AMS).

COMMITTEE ON THE NATIONAL HIGH SCHOOL CONTEST†

W. H. FAGERSTROM, *Chairman* (1959–1962); C. T. SALKIND, Vice-Chairman (1959–1961), H. L. ALDER (1959–1961), WILLIAM ALLAN (1957–1960), SISTER MARY FELICE (1959–1962), E. E. STROCK (1957–1961).

COMMITTEE ON THE PUTNAM PRIZE COMPETITION

RICHARD E. BELLMAN, *Chairman* (1958–1960); L. E. BUSH, Director (1958–1962), IVAN NIVEN (1959–1961), D. E. RICHMOND (1960–1962).

COMMITTEE ON THE UNDERGRADUATE PROGRAM IN MATHEMATICS

R. C. BUCK, *Chairman* (1959–1962); E. G. BEGLE (1959–1963), L. W. COHEN (1959–1961), W. T. GUY, JR. (1959–1962), R. D. JAMES (1959–1961), J. L. KELLEY (1959–1961), J. G. KEMENY (1959–1963), J. C. MOORE (1959–1961), FREDERICK MOSTELLER (1959–1962), H. O. POLLAK (1959–1962), G. B. PRICE (1959–1963), PATRICK SUPPES (1959–1963), HENRY VAN ENGEN (1959–1961), R. J. WALKER (1959–1963), A. D. WALLACE (1959–1961), R. J. WISNER, *ex officio.* (A. W. TUCKER (July 1, 1959–September 1, 1960) to replace J. C. MOORE.)

COMMITTEE ON VISITING LECTURERS

ROTHWELL STEPHENS, *Chairman* (1958–1961); R. C. FISHER (1959–1961), R. E. GASKELL (1959–1961), P. B. JOHNSON (1959–1961), R. E. JOHNSON (1959–1962), R. A. ROSENBAUM (1959–1962), C. L. SEEBECK, JR. (1960–1962).

COMMITTEE TO CONFER WITH A.M.S.

A. E. MEDER, *Chairman;* C. B. ALLENDOERFER, E. G. BEGLE, H. F. BOHNENBLUST, SAUNDERS MACLANE.

EDITORIAL COMMITTEE ON CARUS MONOGRAPHS

R. P. DILWORTH, *Chairman* (1959–1961); HARLEY FLANDERS (1958–1960), E. E. FLOYD (1956–1961), I. I. HIRSCHMAN (1960–1962), IVAN NIVEN (1960–1962).

FINANCE COMMITTEE

E. A. CAMERON (1958–1961), W. B. CARVER (1960–1963), H. L. ALDER, *ex officio,* H. M. GEHMAN, *ex officio.*

† Terms of office of members of this committee expire on September 1 of last year of service listed.

JOINT COMMITTEE ON PLACES OF MEETINGS

G. R. MACLANE, *Chairman* (1958–1960); R. H. BRUCK (1959–1961), WALLACE GIVENS (1960–1962).

JOINT COMMITTEE TO CONSIDER THE SETTING OF WINTER MEETINGS

G. A. HEDLUND, *Chairman;* R. H. BING, J. W. T. YOUNGS.

NOMINATING COMMITTEE FOR 1960

F. A. FICKEN, *Chairman;* P. B. JOHNSON, E. P. VANCE.

PLANNING COMMITTEE FOR A SURVEY OF EUROPEAN MATHEMATICAL EDUCATION

H. F. FEHR, *Chairman;* E. G. BEGLE, SAUNDERS MACLANE, R. E. K. ROURKE.

REPRESENTATIVES OF THE ASSOCIATION

On the A.A.A.S. Cooperative Committee on the Teaching of Mathematics and Science:
P. S. JONES (1960–1962).
On the American Council on Education:
H. L. ALDER, *ex officio*, C. B. ALLENDOERFER, *ex officio*.
On the Conference Board of the Mathematical Sciences:
H. L. ALDER, *ex officio*, C. B. ALLENDOERFER, *ex officio*.
On the Council of the American Association for the Advancement of Science
L. W. COHEN (1959–1960), S. S. CAIRNS (1960–1961).
On the Governing Council of Mu Alpha Theta:
R. B. DEAL, JR. (1958–1960).
On the National Research Council:
W. L. DUREN, Jr. (July 1, 1959–June 30, 1962).
On the U. S. Commission on Mathematical Instruction:
C. B. ALLENDOERFER (July 1, 1959–June 30, 1961), HENRY VAN ENGEN (July 1, 1959–June 30, 1962).

CALENDAR OF FUTURE MEETINGS

Forty-first Summer Meeting, Michigan State University, East Lansing, Michigan, August 29–31, 1960.

Forty-fourth Annual Meeting, Willard Hotel, Washington, D. C., January 25–27, 1961.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 17–18, 1961.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NEW JERSEY
NORTHEASTERN, Wesleyan University, Middletown, Connecticut, November 26, 1960.
NORTHERN CALIFORNIA, San Jose State College, January 14, 1961.
OHIO
OKLAHOMA, Oklahoma City University, October 21–22, 1960.
PACIFIC NORTHWEST
PHILADELPHIA, Swarthmore College, Swarthmore, Pennsylvania, November 26, 1960.
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, University of California, Santa Barbara, March 11, 1961.
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN

Comparison of program tapes for a vibrational problem expressed in DYANA language, in algebraic-oriented language, and in the basic machine language.

One of a series

Higher Education for Computers

"Let's put the computer in at the start of a problem, rather than just having it buzz through the computations."

This is the approach being taken by computer specialists at the General Motors Research Laboratories as they explore ways of giving large-scale digital computers a greater role in the solution of problems. The object is to "teach" computers to apply the same rules men use in formulating, analyzing, and solving questions of modern science and engineering.

A recent outgrowth of this work is DYANA, GM Research's new automatic analysis and programming system. DYANA is one of the first computer systems to "understand" declarative statements. For a large class of dynamic problems, the engineer can simply describe his physical system to the computer. The computer figures out how to handle it.

For the solution, DYANA automatically directs the computer to prepare a mathematical model of the system, to write its own program for solving the model, then to execute the program and compute the desired answers.

The higher education of computers currently involves studies in symbol manipulation, problem-oriented languages, character and pattern recognition, and engineering simulation.

Such advanced computer concepts are giving General Motors professional people more time for creative engineering and research—time to explore ideas and to develop "more and better things for more people."

General Motors Research Laboratories
Warren, Michigan

Careers in

Mathematics

Vitro's increased activities in the field of Operations Research have created career opportunities for men with these interests and qualifications:

MATHEMATICAL STATISTICIANS

MS or PhD for conducting and consulting on analytical studies in a wide variety of applications, including information theory, weapons systems analysis, experimental design, data treatment. Should be familiar with some of the following—Monte Carlo procedures, Markov processes, decision theory, operations research, and have had 3-5 years industrial experience in implementing these techniques. Position is in the Information Analysis Group.

OPERATIONS RESEARCH ANALYSTS

MA or PhD in Mathematics, Statistics or Physics. Conduct and direct operations research studies, principally in the areas of weapons systems evaluation, ballistic missile defense, anti-submarine warfare and electronic countermeasures. Should have experience in some of the following areas: applications of game theory, linear programming, Monte Carlo techniques, queueing theory and model construction.

► *Our modern laboratory is located in a suburban area with easy access to the cultural and educational facilities of metropolitan New York and New Jersey. Liberal benefits include a tuition refund plan and relocation allowances.*

Send resume and salary requirements to Mr. S. Roberts.

Vitro LABORATORIES

Division of Vitro Corp. of America

200 Pleasant Valley Way, West Orange, New Jersey

Opportunities in Computer Programming

Making a machine part of a system

To start with the raw computer components and shape them into an effective computer system—this is Systems Programming at IBM. It's a field of work that offers rich opportunities for achievement to experienced programmers.

In Systems Programming, you may have the chance to work with advanced computers even before they are commercially released. You may be working in one or more of these areas:

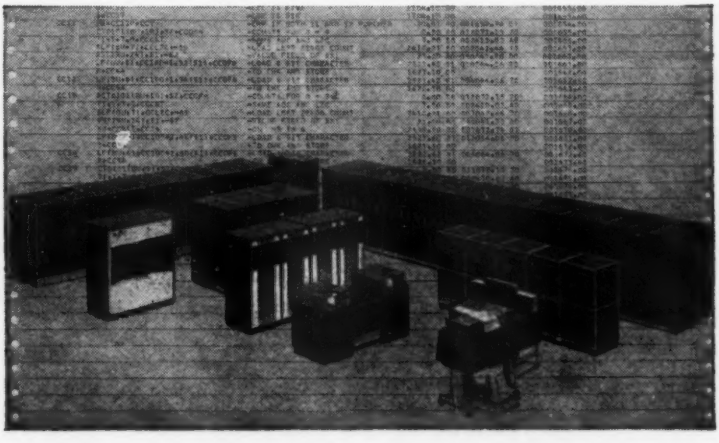
- Creating master control programs
- Establishing automatic operating systems
- Providing for exception cases (interrupt system)
- Determining termination procedures
- Programming advanced computer consoles which can be used to adjust parameters.

If you are a qualified programmer or mathematician and interested in joining professional people working in a professional atmosphere—with ample computer time to test your ideas—write to:

Mr. R. L. Jerue, Dept. 510R
IBM Corporation
Box 390
Poughkeepsie, N. Y.

IBM®

INTERNATIONAL BUSINESS MACHINES CORPORATION



NAIVE SET THEORY

by PAUL R. HALMOS, University of Chicago

The University Series in Undergraduate Mathematics

This brief, informal exposition of set theory is written from the viewpoint of a prospective mathematician whose main interest is not in the foundations of mathematics but in such concrete mathematical objects as groups, integrals, and manifolds. With a minimum of philosophical discourse and logical formalism, Professor Halmos tells the beginning graduate student what he considers "the basic set-theoretic facts of life."

1960 104 pages
$3.50

D. VAN NOSTRAND COMPANY, INC.
120 ALEXANDER ST. PRINCETON, N. J.

THE CARUS MATHEMATICAL MONOGRAPHS

This new Carus Monograph was published in May 1960:
MONOGRAPH 13: A Primer of Real Functions, by Ralph P. Boas Jr.

Each member of the Association may purchase one copy of each monograph at the special price of $2.00. *Orders should be addressed to:*

HARRY M. GEHMAN, *Treasurer*
Mathematical Association of America
University of Buffalo
Buffalo 14, New York

Additional copies of Monograph 13 for members and copies for non-members may be purchased at $4.00 from:

JOHN WILEY AND SONS
440 Fourth Street, New York 16, New York

NEW ADDITIONS TO VAN NOSTRAND'S

UNIVERSITY SERIES IN HIGHER MATHEMATICS

Commutative Algebra

Volume II

by Oscar Zariski, *Harvard University;* and Pierre Samuel, *University of Clermont-Ferrand.*

Devoted to valuation theory, polynomial and power series rings, and local algebra, this volume completes the first systematic treatment of commutative algebra since Krull's monograph of 1935. The algebro-geometric sources and applications of the purely algebraic material are constantly emphasized.

400 pp. $9.75

Probability Theory

2nd edition

by Michel Loève, *University of California*

The second edition of this highly respected work features two entirely new chapters which carry the exposition into an investigation of random functions and Markov processes. Professor Loève has also made a large number of minor changes in the ten original chapters.

650 pp. about $15.00

D. VAN NOSTRAND COMPANY, INC.

PRINCETON, NEW JERSEY

PROFESSIONAL OPPORTUNITIES IN MATHEMATICS

Fourth Edition ***July 1959*** ***Reprinted 1960***

The report of a committee of the MAA

CONTENTS:

The Teacher of Mathematics; Opportunities in Mathematical and Applied Statistics; The Mathematician in Industry; Mathematicians in Government; Opportunities in the Actuarial Profession; Non-Academic Employment of Mathematicians; References for Further Reading.

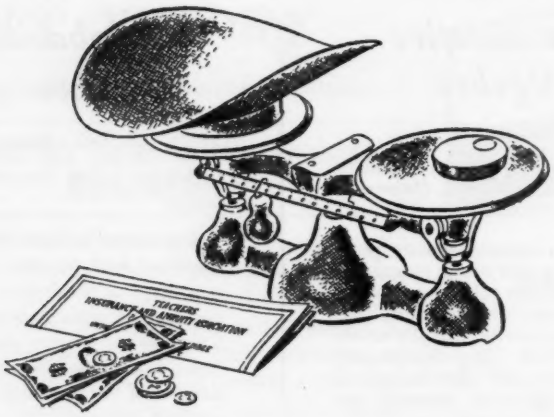
24 pages, paper covers

25¢ for single copies; 20¢ each for orders of five or more.

Send orders to:

HARRY M. GEHMAN, *Treasurer*
Mathematical Association of America
University of Buffalo
Buffalo 14, New York

WEIGH YOUR FAMILY'S FUTURE....



Against The Low Cost of TIAA Life Insurance

A little premium buys a lot of family security. For example, a man 34 years of age can provide $20,000 of protection on the 10-Year Term Plan for only $82.40* a year.

TIAA can offer insurance at low cost for several reasons: (1) it is a nonprofit company established by Carnegie organizations to serve the educational world; (2) it employs no agents and pays no commissions; and (3) occupational hazards are almost non-existent in the college world.

Employees of colleges, universities, nonprofit educational and research institutions and private schools are eligible to apply for TIAA insurance.

Many TIAA plans are available to help you provide the protection and security you want for your family and to help keep your insurance costs within your budget. To get your copy of *The Life Insurance Guide*, which describes them, simply fill out the coupon below and mail it to us. If you give us your date of birth and the ages of your dependents, we will be glad to illustrate specific TIAA plans for you.

No agent will call since TIAA employs none. Your information will be sent by mail.

* $132.20 annual premium less $49.80 cash dividend paid at end of year. Future dividend amounts cannot be guaranteed, of course.

TIAA
730 Third Avenue
New York 17, N.Y.

Please send me a Life Insurance Guide and an illustration of low-cost protection at my age. F

Name_____ Date of Birth_____

Address_____

Ages of Dependents_____

Employing Institution_____

TIAA

TEACHERS INSURANCE AND ANNUITY ASSOCIATION

□ A

□ I

□ I

□ C

□ C

□ C

□ S

□

□

A Checklist of Recently Published and In-Press McGraw-Hill Textbooks

☐ AGNEW—DIFFERENTIAL EQUATIONS, *New Second Edition*, 512 pages, $7.50

☐ BRYANT—STATISTICAL ANALYSIS, 320 pages, $6.50

☐ BUTLER AND WREN—TEACHING OF SECONDARY MATHEMATICS, *New Third Edition*, Ready in September

☐ CHURCHILL—COMPLEX VARIABLES AND APPLICATIONS, *New Second Edition*, 297 pages, $6.75

☐ GARVIN—INTRODUCTION TO LINEAR PROGRAMMING, Ready in August

☐ GOERTZEL AND TRALLI—SOME MATHEMATICAL METHODS OF PHYSICS, 300 pages, $8.50

☐ KELLS—ELEMENTARY DIFFERENTIAL EQUATIONS, *New Fifth Edition*, 318 pages, $6.25

☐ SMITH—ELECTRONIC DIGITAL COMPUTERS, 443 pages, $12.00

☐ SPARKS—A SURVEY OF BASIC MATHEMATICS: A Text and Workbook for College Students, 336 pages, $3.95

Send for Copies on Approval

McGRAW-HILL BOOK COMPANY, Inc.

330 West 42nd Street New York 36, N. Y.

Published in May, this concise, (159 page) paperbound book lives up to its name; it examines the basic mathematics of Boolean algebra and explores applications, especially those in electrical engineering. As the author, Franz E. Hohn, Associate Professor of Mathematics at the University of Illinois, mentions in his Preface, ". . . anyone who has had high school mathematics and physics and who has a little courage should be able to read the entire volume with profit."

APPLIED BOOLEAN ALGEBRA: An Elementary Introduction

by Franz E. Hohn

Including more mathematics and theory than general surveys that discuss this topic, APPLIED BOOLEAN ALGEBRA also serves as a useful and informative review of fundamentals. Even the more advanced student will find the work sound and the treatment of concepts rigorous and consistent.

Fully and independently organized for each of three major applications, the book covers: Boolean algebra as a model of combinational relay circuitry, as a model of propositional logic, and the Boolean algebra of the subsets of a set. The two appendices contain material on the binary system of numeration and the physical aspects of switching elements.

Included in a series of mathematics texts under the general editorship of Carl B. Allendoerfer, this book is available in paperback. Probably $2.50.

Also by Franz E. Hohn . . .

ELEMENTARY MATRIX ALGEBRA

"The link which makes such a small text useful in so many fields is the growing tendency to use matrix algebra as a mathematical tool. . . . Specific examples, some completely worked out, add to the value of the book. The exercises vary from simple to complex and, in general, represent definite applications in a particular field, although their solution does not depend on a knowledge of that field."—*The Journal of the Franklin Institute* 1959, 305 pages, $7.50

The Macmillan Company

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN, U.S.A.